1990

Use of continuation methods for kinematic synthesis and analysis

Thiagaraj Subbian
Iowa State University

Order Number 9110571

Use of continuation methods for kinematic synthesis and analysis

Subbian, Thiagaraj, Ph.D.

Iowa State University, 1990

U·M·I
300 N. Zeeb Rd.
Ann Arbor, MI 48106

Use of continuation methods for kinematic synthesis and analysis

by

Thiagaraj Subbian

A Dissertation Submitted to the

Graduate Faculty in Partial Fulfillment of the

Requirements for the Degree of

DOCTOR OF PHILOSOPHY

Major: Mechanical Engineering

Approved:

Signature was redacted for privacy.

In Charge of Major Work

Signature was redacted for privacy.

For the Major Department

Signature was redacted for privacy.

For the Graduate College

Iowa State University
Ames, Iowa
1990

TABLE OF CONTENTS

INTRODUCTION

Synthesis and analysis of mechanisms have been accomplished in the past by graphical and analytical methods. Graphical methods are mainly useful for planar mechanisms. The success of these methods is limited to a few types of mechanisms involving only a few precision points. Furthermore, all possible solutions may not be obtained graphically. Closed-form analytical solutions are possible but are difficult to obtain for more than a few precision points. Due to the above reasons, application of numerical methods is inevitable.

Newton-Raphson, steepest descent and generalized reduced gradient are the commonly used numerical methods for kinematic synthesis and analysis. These procedures, due to their inadequacies, limit the number of precision points or the type of mechanism to be synthesized. In the above methods, the user has to come up with initial solution estimates, and the reliability of the procedure depends on the choices made. This need imposes a particularly difficult requirement in the synthesis problem because there is usually no logical basis for estimating starting values for link dimensions. Furthermore, when multiple solutions exist, these methods do not offer a methodical procedure for finding them. Therefore, this calls for a numerical procedure that does not need initial guesses and provides all the solutions.

Continuation methods fall in this category for polynomial equations. The kinematic analysis and synthesis problems can be easily expressed in polynomial form and so the procedure is particularly useful for these applications. Also, the method can be used to solve n equations in $(n+1)$ unknowns, which could be used to generate the Burmester curves in kinematic synthesis, and the $(n+1)$th variable can be used as an independent variable for analysis.

Dimensional synthesis is often classified as function generation, path generation, motion generation, path generation with timing, or motion generation with timing. To carry out the synthesis tasks, different techniques, such as precision position synthesis [Erdman, 1981], selective precision synthesis [Kramer and Sandor, 1975], and least squares [Sarkisyan et al., 1973] method have been applied. In this dissertation, the above synthesis tasks are carried out for precision position synthesis using continuation. One popular approach to kinematic synthesis is to break up the linkage into dyads and triads and to design them independently [Sandor and Erdman, 1984, Suh and Radcliffe, 1978, and Erdman, 1981]. Here, four-bar function generation, five-bar path generation, and six- and eight-bar motion generation with prescribed timing mechanisms are designed using this approach.

Application of a type of continuation for kinematic synthesis was first carried out by Roth and Freudenstein [1963]. They termed their technique the *bootstrap* method. Only recently has further effort been focused on continuation methods for mechanism synthesis. Researchers such as Morgan and Sommese [1989], as well as Garcia and Zangwill [1981], have made continuation methods more efficient and easier to implement. These advancements combined with rapid developments in computer technology have made the application of continuation methods for kinematic synthesis feasible and highly desirable.

A detailed description of homotopy continuation methods is provided in Morgan [1986a and 1987], and Garcia and Zangwill [1981]. The number of paths to be tracked in solving a system of polynomial equations is given by Morgan [1987]. The solutions obtained can be finite or infinite in value. Projective transformations can be used to transform solutions at infinity to be finite in value [Morgan, 1986b and 1987]. Some of the extraneous paths which result in solutions at infinity can be eliminated by taking advantage of the m-homogeneous structure of the equations being solved. The procedure to identify the multi-homogeneous structure and to apply projective transformations is discussed by Morgan and Sommese [1987a and 1987b]. Morgan and Sommese [1989] and Wampler et al. [1990a] outline the various homotopies and discuss their relative merits and deficiencies. The procedure to eliminate coefficient independent solutions at infinity for a m-homogeneous system is also provided in the above references.

This dissertation is composed of six major parts, which present various aspects of the application of continuation methods to kinematic synthesis and analysis. The material in these six parts has been extracted from papers which have already been published, or which have been submitted for publication and are at present in various stages of review [Subbian and Flugrad, 1989a, 1989b, 1990a, 1990b, 1991, and Subbian et al. 1991]. Part I, entitled *Synthesis of Motion Generating Mechanisms by a Continuation Method* deals with the synthesis of four-bar motion generation mechanisms for four and five prescribed precision positions. Closed-form solutions are available for the four position problem [Sandor and Erdman, 1984]; however, some sort of numerical procedure is required for five prescribed positions. Dyad Burmester curves are generated for the four position problem and four finite solutions are obtained for the five position problem.

Part II, entitled *Four-Bar Path Generation Synthesis by a Continuation Method*, involves the design of a four-bar path generation mechanism for five prescribed positions. A complete solution cannot be obtained for this problem in closed form or by other numerical procedures. Hence, a continuation was used involving 256 starting points in 1-homogeneous form, and all possible solutions were obtained systematically.

Part III, entitled *Five Position Triad Synthesis With Applications to Four- and Six-Bar Mechanisms*, extends the continuation procedure to six-bar mechanisms by considering the mechanisms in terms of a triad and two dyads. The triad is designed for five precision positions for motion generation with prescribed timing. As the number of variables exceed the number of equations by two, solution curves are obtained. The dyads are designed independently, and the six-bar motion generation mechanism with prescribed timing is obtained.

Part IV, entitled *Six and Seven Position Triad Synthesis Using Continuation Methods*, extends the five position triad synthesis of part III to six and seven positions. Seven is the maximum number of positions possible; thus a finite solution set is obtained for this case. For the six position case, triad Burmester curves are obtained. A secant homotopy was used, reducing the number of paths to be traced from 32 to 15 for the six position problem, and from 64 to 17 for the seven position problem. An eight-bar motion generation mechanism with prescribed timing, and a geared five-bar path generating mechanism were designed using the triads.

In Part V, which is entitled *Use of Continuation Methods for Kinematic Synthesis*, the available path reduction schemes are outlined by means of planar and spatial mechanism examples. Seven position triad synthesis and revolute-spherical dyad synthesis problems are considered. Implementation of continuation methods to solve these problems is discussed in detail, and path reduction techniques which take advantage of the m-homogeneous structure of the equations and which utilize parameter and secant homotopies are described. A step by step procedure is outlined.

Part VI, which is titled *Robot Trajectory Planning by a Continuation Method*, concerns the application of continuation methods for kinematic analysis. Trajectory planning of 3-R and 6-R robots is carried out by solving n equations in $(n + 1)$ unknowns. The joint variables (θ's) are the n variables and the parameter defined along the trajectory of the robot is considered as the $(n + 1)$th variable. The n equations are obtained by expressing the cartesian (x, y, z) coordinates of the tip of the robot as a function of the link parameters and joint variables. Knowing the inverse kinematic solution for the robot at the initial position, and the path to be traced, the inverse kinematic solution along the path is computed by solving the n equations in $(n + 1)$ unknowns.

PART I.

SYNTHESIS OF MOTION GENERATING MECHANISMS BY A CONTINUATION METHOD

INTRODUCTION

Four-bar motion generating mechanisms can be synthesized by graphical methods, by algebraic formulation, or by complex number formulation. Of these, complex number formulation, as presented by Erdman [1981] and Chase et al. [1985] is simple to implement on a digital computer and has gained a great deal of popularity. This approach is used here to derive the kinematic equations for a motion generation problem, and the resulting equations are solved using continuation methods.

Four-bar motion generating mechanisms can be designed for a maximum of five prescribed positions, but closed-form solutions are possible only when four positions or less are specified [Sandor and Erdman, 1984]. Thus, numerical methods are essential to solve the five position synthesis case. A continuation method approach is presented here towards that end. The development of Burmester curves for the four position case again using continuation is also addressed.

The continuation method is a mathematical procedure which provides all possible solutions for systems of polynomial equations. No initial solution estimates are needed, and the method can be used to solve a system of n equations in either n or $(n+1)$ unknowns. The latter problem results in solution curves for the infinities of possible solutions. These are a few of the advantages of the method being proposed over others currently employed. The continuation method has already been applied to solve path generation synthesis by Subbian and Flugrad [1989] and by Roth and Freudenstein [1963]; here it is extended to motion generation mechanisms.

A brief sketch of the method is provided in sections to follow, along with the development of equations for a general motion generation problem. The application of this technique for solving four and five position problems is presented in detail, and an example is worked for each case.

SOLVING n EQUATIONS IN n UNKNOWNS

To solve a system of n polynomial equations in n unknowns, say $\mathbf{F}(z) = 0$, a simple set of n equations in the same unknowns, say $\mathbf{G}(z) = 0$, is first assumed. The solutions of the assumed system are known and we are interested in determining the solutions of the given system. The system $\mathbf{F}(z) = 0$ is referred to as the target system and $\mathbf{G}(z) = 0$ is referred to as the start system. The start system is chosen to be of the same degree as the target system. A homotopy function $\mathbf{H}(z,t) = 0$ is then defined using the homotopy parameter t, $\mathbf{H}(z,t) = t\mathbf{F}(z) + (1-t)\mathbf{G}(z) = 0$. When $t = 0$, the homotopy functions reduce to the start system and when $t = 1$ they represent the target system. Thus, by increasing t from 0 to 1, and by solving the intermediate subproblems along the way, the solutions to the original system of equations are found. For tracking the variables as the parameter t is increased either Cramer's rule or Gaussian elimination methods can be used. A detailed description of the method is provided in the parts to follow as well as Morgan [1987] and Subbian and Flugrad [1989].

SOLVING n EQUATIONS IN (n+1) UNKNOWNS

Let $\mathbf{E}(x) = 0$ be a system of n polynomial equations in $(n+1)$ variables, x. Since the number of unknowns exceeds the number of equations, a set of curves is obtained as the solution. If one point on a solution curve is known, that particular curve can be traced using a continuation method. Therefore, all the solution curves can be traced if a point on each is known. These points can be obtained by assuming an arbitrary value for one of the unknowns and then using the continuation procedure described in the previous section to solve the resulting n equations in n unknowns.

Assuming that a point on a specific curve has been found, one can determine the curve by first evaluating the extended Jacobian matrix for the system:

$$DE = \begin{bmatrix} \frac{\partial E_1}{\partial x_1} & \frac{\partial E_1}{\partial x_2} & \cdots & \frac{\partial E_1}{\partial x_{n+1}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial E_n}{\partial x_1} & \frac{\partial E_n}{\partial x_2} & \cdots & \frac{\partial E_n}{\partial x_{n+1}} \end{bmatrix}$$

First order differential equations of the unknowns with respect to a path parameter s are then obtained from the Jacobian matrix as $dx_i/ds = C(-1)^{i+1}det(DE_{[i]})$ with $i = 1, 2, ..., n+1$ and, $DE_{[i]}$ as the Jacobian matrix with the ith column deleted. C is a randomly chosen real constant.

Therefore, using the initial points on the curves, the differential equations can be integrated to obtain the solution curves, by first choosing $C > 0$ and then $C < 0$ [Morgan, 1981].

DEVELOPMENT OF EQUATIONS FOR THE MOTION GENERATION PROBLEM

The four-bar mechanism as defined by two dyads is shown in Fig. 1. For motion generation synthesis, it is sufficient to consider one dyad, as given in Fig. 2, in two finitely separated positions. The loop closure equation for this dyad can be written as

$$\mathbf{W}(e^{i\phi_j} - 1) + \mathbf{Z}(e^{i\gamma_j} - 1) \;=\; \delta_{\mathrm{j}} \tag{1}$$

W and **Z** are the unknown complex link vectors, and ϕ_j is the unknown jth angular displacement of the **W** link from its initial configuration. The specified jth complex displacement of a point on the coupler from its initial position is given by δ_{j}, and the change in coupler orientation, which is also provided, is represented by γ_j.

If **Z**, **W** and δ_{j} are replaced in terms of real and imaginary components and the exponentials are expanded using Euler's equation, Eq. (1) reduces to

$$(W_x + iW_y)(cos\phi_j - 1 + isin\phi_j) + (Z_x + iZ_y)(cos\gamma_j - 1 + isin\gamma_j) = \delta_{jx} + i\delta_{jy} \tag{2}$$

By treating $cos\phi_j$ and $sin\phi_j$ as the two independent variables $C\phi_j$ and $S\phi_j$, respectively, and separating out the real and imaginary terms, the following expressions are obtained:

$$W_x(C\phi_j - 1) - W_yS\phi_j + Z_x(cos\gamma_j - 1) - Z_ysin\gamma_j \;=\; \delta_{jx} \tag{3}$$

$$W_xS\phi_j + W_y(C\phi_j - 1) + Z_xsin\gamma_j + Z_y(cos\gamma_j - 1) \;=\; \delta_{jy} \tag{4}$$

Eliminating $S\phi_j$ and $C\phi_j$ from Eqs. (3) and (4) and using the constraint equation $C\phi_j^2 + S\phi_j^2 = 1$ [Subbian and Flugrad, 1989], one obtains:

$$F_{j-1} = 2A_jsin\gamma_j - 2B_jcos\gamma_j + 2B_j + D_j = 0 \tag{5}$$

where $j = 2$ to n, and

$$\begin{aligned}
A_j &= W_x Z_y - W_y Z_x + Z_y \delta_{jx} - Z_x \delta_{jy} \\
B_j &= Z_x^2 + Z_y^2 + W_x Z_x + W_y Z_y + Z_x \delta_{jx} + Z_y \delta_{jy} \\
D_j &= 2W_x \delta_{jx} + 2W_y \delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2
\end{aligned}$$

In the above expression, j represents the individual displacements from an initial position. As there are only four unknowns in Z_x, Z_y, W_x and W_y, a maximum of four equations written for five prescribed positions can be solved. Free choices in the link length vectors are made if the synthesis is carried out for fewer than five positions. However, if less than five positions are used, it is easier to work with a set of redefined variables as outlined in the following section.

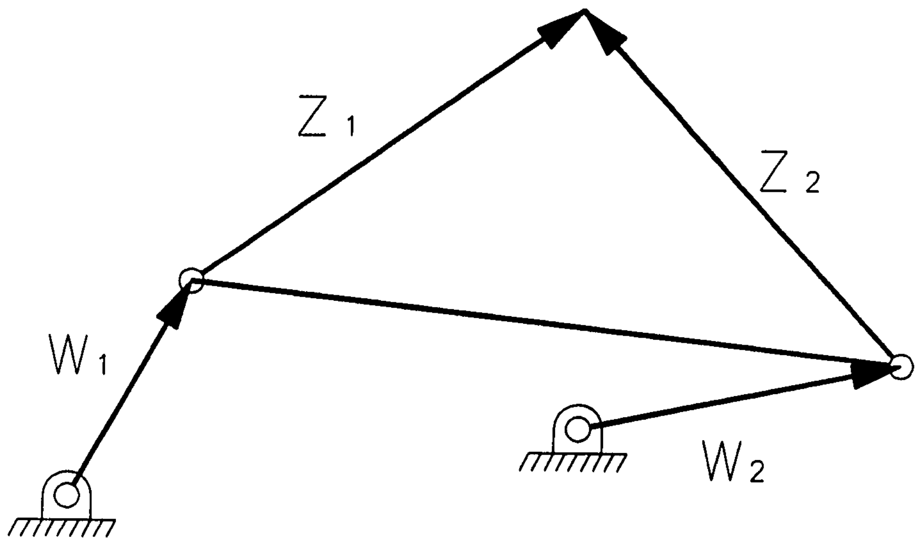

Figure 1: Four-bar mechanism defined by two dyads

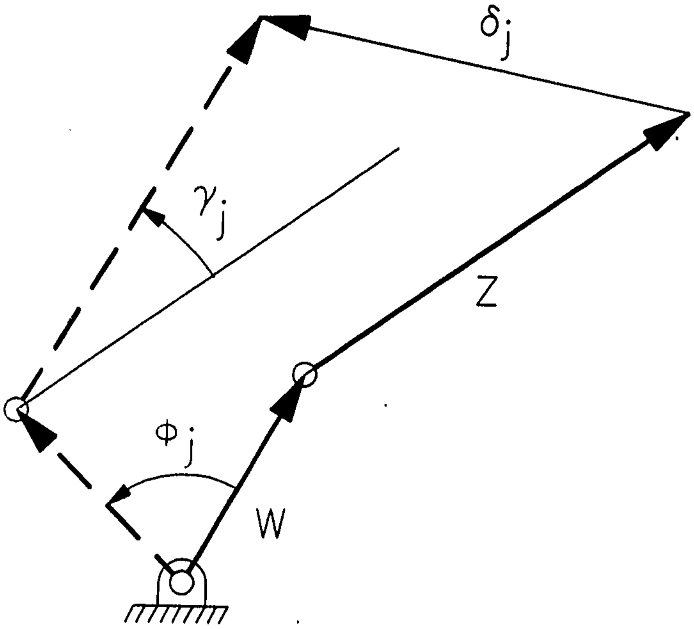

Figure 2: A dyad in two finitely separated positions

FOUR POSITION SYNTHESIS

For a four position synthesis problem it is helpful to replace the unknowns Wx and Wy by new variables R and θ. These variables locate the radius and orientation of the fixed pivot with respect to the coupler point in its initial configuration, as indicated in Fig. 3. Wx and Wy can be eliminated from Eq. 5 using the relationships:

$$\begin{aligned} Zx + Wx &= -R\cos\theta \\ Zy + Wy &= -R\sin\theta \end{aligned} \tag{6}$$

On elimination of Wx and Wy, F_{j-1} is reduced to the following expression:

$$F_{j-1} = a_j RZx + b_j RZy + c_j Zx + d_j Zy + e_j R + f_j \tag{7}$$

where, $j = 2, 3, 4$, and

$$\begin{aligned} a_j &= 2\sin\theta\sin\gamma_j + 2\cos\theta\cos\gamma_j - 2\cos\theta \\ b_j &= -2\cos\theta\sin\gamma_j + 2\sin\theta\cos\gamma_j - 2\sin\theta \\ c_j &= -2\delta_{jy}\sin\gamma_j - 2\delta_{jx}\cos\gamma_j \\ d_j &= 2\delta_{jx}\sin\gamma_j - 2\delta_{jy}\cos\gamma_j \\ e_j &= -2\delta_{jy}\sin\theta - 2\delta_{jx}\cos\theta \\ f_j &= \delta_{jx}^2 + \delta_{jy}^2 \end{aligned}$$

For a four position motion generation problem δ_2, δ_3, δ_4, γ_2, γ_3 and γ_4 are specified to obtain three equations in four unknowns. Thus, by specifying one of the unknowns, Eqs. (7) can be solved. Once points on the solution curves are obtained, the curves can be traced using the continuation procedure discussed earlier to solve n equations in $(n+1)$ unknowns. These solution procedures are utilized in this example to generate the Burmester curves.

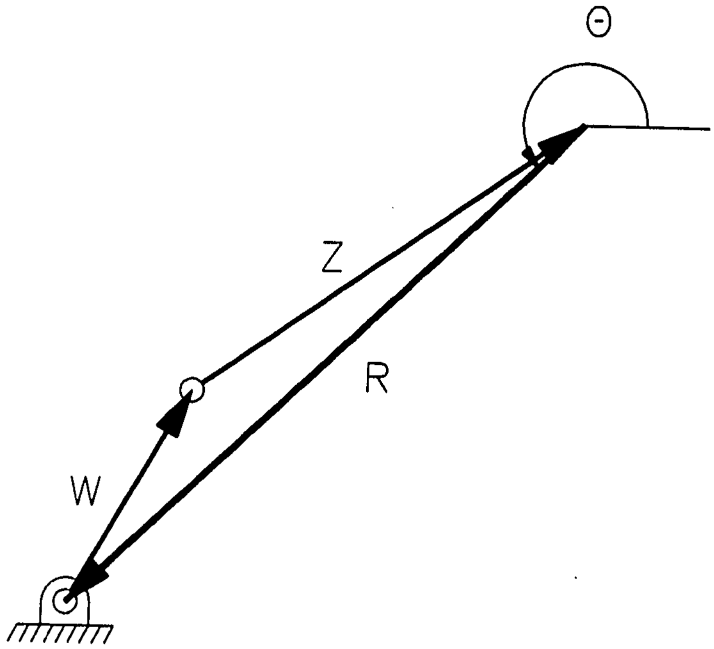

Figure 3: The dyad in terms of defined variables R and θ

Determination of initial points on the curve

To determine solution points on the Burmester curves, an arbitrary value is assigned to one of the variables. Here, θ is chosen and Zx, Zy and R are evaluated. The equations to be solved, obtained from Eqs. (7), are:

$$\begin{aligned}
F_1 &= a_2 ZxR + b_2 ZyR + c_2 Zx + d_2 Zy + e_2 R + f_2 = 0 \\
F_2 &= a_3 ZxR + b_3 ZyR + c_3 Zx + d_3 Zy + e_3 R + f_3 = 0 \\
F_3 &= a_4 ZxR + b_4 ZyR + c_4 Zx + d_4 Zy + e_4 R + f_4 = 0
\end{aligned} \tag{8}$$

R can easily be eliminated from Eqs. (8), which results in equations,

$$E_{j-2} = TR1_j Zx^2 + TR2_j ZxZy + TR3_j Zy^2 + TR4_j Zx + TR5_j Zy + TR6_j \tag{9}$$

where $j = 3$, 4, and

$$\begin{aligned}
TR1_j &= a_2 c_j - c_2 a_j \\
TR2_j &= a_2 d_j - d_2 a_j + b_2 c_j - c_2 b_j \\
TR3_j &= b_2 d_j - d_2 b_j \\
TR4_j &= a_2 f_j - f_2 a_j + e_2 c_j - c_2 e_j \\
TR5_j &= b_2 f_j - f_2 b_j + e_2 d_j - d_2 e_j \\
TR6_j &= e_2 f_j - f_2 e_j
\end{aligned}$$

Each of Eqs. (9) are of degree two, and thus the total degree of the system is four. Therefore, there is a maximum of four solutions. To solve the system, a start system in Zx and Zy is assumed. The degree of this system should be the same as that of the original set of equations. We can use:

$$G_1 = C_{11} Zx^2 - C_{12} = 0$$

$$G_2 = C_{21}Z_y^2 - C_{22} = 0 \tag{10}$$

where C_{11}, C_{12}, C_{21} and C_{22} are random complex constants.

The homotopy functions can now be written by combining Eqs. (9) and Eqs. (10) as

$$\begin{aligned} H_1 &= E_1 t + (1-t)G_1 = 0 \\ H_2 &= E_2 t + (1-t)G_2 = 0 \end{aligned} \tag{11}$$

The partial derivatives of the homotopy functions with respect to Z_x, Z_y and t are determined to form the extended Jacobian matrix.

$$DH = \begin{bmatrix} \partial H_1/\partial Z_x & \partial H_1/\partial Z_y & \partial H_1/\partial t \\ \partial H_2/\partial Z_x & \partial H_2/\partial Z_y & \partial H_2/\partial t \end{bmatrix}$$

The first order derivatives dZ_x/dt and dZ_y/dt are determined as follows:

$$dZ_x/dt = det(DH_{[1]})/Den$$

$$dZ_y/dt = -det(DH_{[2]})/Den$$

where $DH_{[j]}$ is the Jacobian matrix with the jth column deleted, and

$$Den = (-1)^{n+2} det(DH_{[n+1]})$$

The solution to the original problem is obtained, therefore, by integrating the above differential equations between the limits of $t = 0$ and $t = 1$. The solutions to the start system in Eqs. (10) are the starting points of the integration.

To generate the Burmester curves

Once points on the Burmester curves are obtained using the procedure outlined above, the next step is to generate the curves. In order to do this Eqs. (8), which involve all four unknowns, are used. The first step is to determine the Jacobian for the system in the following form:

$$DF = \begin{bmatrix} \partial F_1/\partial Z_x & \partial F_1/\partial Z_y & \partial F_1/\partial R & \partial F_1/\partial \theta \\ \partial F_2/\partial Z_x & \partial F_2/\partial Z_y & \partial F_2/\partial R & \partial F_2/\partial \theta \\ \partial F_3/\partial Z_x & \partial F_3/\partial Z_y & \partial F_3/\partial R & \partial F_3/\partial \theta \end{bmatrix}$$

From this Jacobian matrix the first order derivatives of the variables with respect to the solution curve parameter or path parameter, s, can be evaluated. These first order differential equations are then integrated with the points obtained above as initial conditions, to trace the Burmester curves.

Four position synthesis example

A four-bar mechanism was designed using continuation method to satisfy the coupler displacement conditions given in Table 1. The angular orientation, θ, of the fixed pivot with reference to the coupler point was chosen to be 0 degrees, and the unknowns, Z_x and Z_y, were determined. A total of four real solutions (dyads) were obtained. These are listed in Table 2.

Of the four solutions, dyad 4 is not usable since R is of the form $0/0$. This means that R can take on any value. Because the first of Eqs. (8) was used to eliminate R, the remaining two precision points (3 and 4) will fix a specific value for R. The other three dyads can be combined to form three different four-bar mechanisms. Combination of dyads 1 and 3 gave a double rocker mechanism with no defects. This

Table 1: Prescribed precision points for four position synthesis example

Precision Point	δ_x	δ_y	γ deg.
2	2	0	30
3	7	5	45
4	10	8	30

Table 2: Dyads satisfying the prescribed conditions for $\theta = 0$ degrees (four position synthesis example)

No.	Z_x	Z_y	R
1	-4.8028	-1.7115	17.1872
2	15.8236	-2.8823	-10.5605
3	5.9630	6.1047	-0.3800
4	-1.0	-3.7321	0/0

mechanism in its four prescribed positions along with the coupler curve is shown in Fig. 4. All the remaining combinations are defective and so are not shown.

In order to generate the Burmester curves, the solutions obtained above were used as starting points and the solution paths traced. Among the three good dyad pairs obtained, dyads 2 and 3 fell on the same Burmester curve and so resulted in the same branch. Dyad 1 provided a unique trace. The resultant Burmester curves of fixed pivots and moving pivot locations with respect to the coupler in its initial position are shown in Fig. 5.

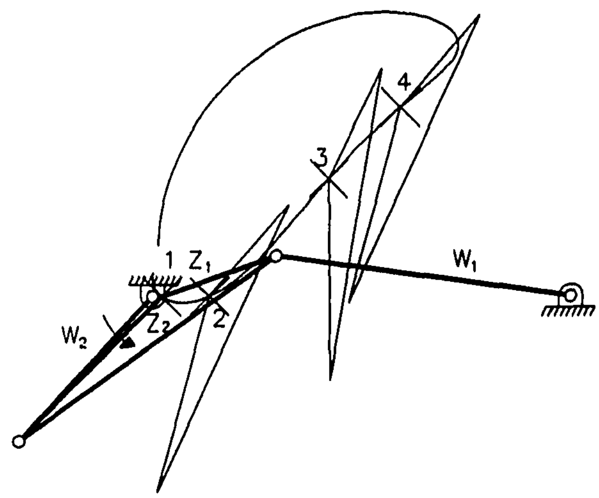

Figure 4: A four-bar mechanism combining dyads 1 and 3 of Table 2

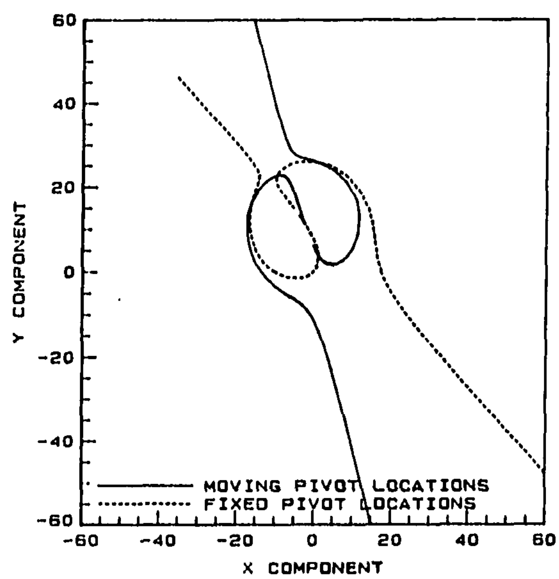

Figure 5: Burmester curves obtained for the four position synthesis example

FIVE POSITION SYNTHESIS

For a five position synthesis problem, R_x and R_y are used to replace $R\cos\theta$ and $R\sin\theta$, respectively, in Eq. (6). The above expressions are used to eliminate W_x and W_y from Eq. (5) to obtain the following relationship:

$$\begin{aligned} F_{j-1} &= 2(R_yZ_x - R_xZ_y + Z_y\delta_{jx} - Z_x\delta_{jy})\sin\gamma_j \\ &\quad -2(Z_x\delta_{jx} + Z_y\delta_{jy} - R_xZ_x - R_yZ_y)(\cos\gamma_j - 1) \\ &\quad -2(R_x + Z_x)\delta_{jx} - 2(R_y + Z_y)\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \end{aligned} \tag{12}$$

where, $j = 2, 3, 4, 5$ for five position synthesis.

Each of the above equations are of degree two in the unknowns Z_x, Z_y, R_x and R_y, and so the total degree of the system is 16 (2^4). This system of four second degree equations in four unknowns can be solved using the procedure outlined earlier. There are a total of 16 paths originating from the 16 solutions to the start system of equations. The solutions are either real or complex, or they diverge to infinity. The real solutions thus determined provide dyads satisfying the prescribed conditions. The combination of pairs of these dyads result in the desired four bar motion generating mechanisms.

Five position synthesis example

Four-bar mechanisms were designed using continuation for the conditions listed in Table 3. A total of four real solutions were obtained as given in Table 4. The remaining solutions were at infinity. The four real solutions (dyads) can be combined to form six four-bar mechanisms satisfying the prescribed conditions. Of the various possible combinations, the combination of dyads 2 and 3 was defective. All the remaining pairs, however, were acceptable, totaling two crank-rocker mechanisms and three double-rocker mechanisms. These mechanisms, along with the resulting coupler curves, precision points and coupler orientations are given in Figs. 6, 7, 8, 9 and 10.

Table 3: Prescribed precision points for five position

Precision Point	δ_x	δ_y	γ deg.
2	-0.6331	-0.5449	12.65
3	-2.0713	-2.3566	42.65
4	-2.5510	-3.5456	57.65
5	-2.7720	-4.5210	67.65

Table 4: Dyads satisfying the prescribed conditions for the five

No.	Z_x	Z_y	R_x	R_y
1	-1.8189	-0.9641	2.5111	-5.0583
2	-0.7807	3.9692	-0.2988	-5.0377
3	-2.5263	4.4872	2.5572	-4.1503
4	-2.0962	3.4017	2.0600	-4.4665

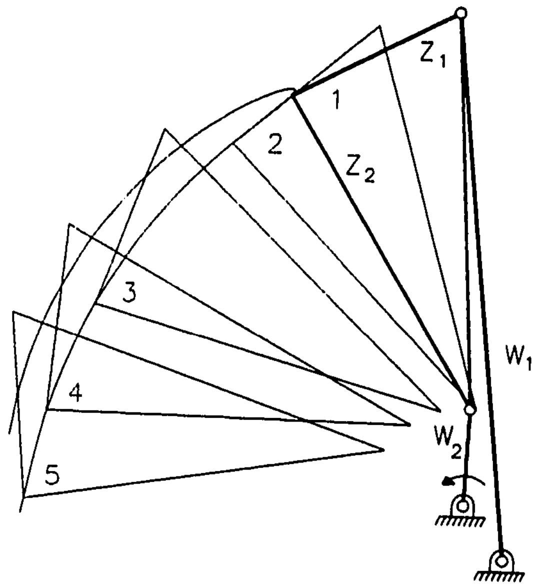

Figure 6: A mechanism combining dyads 1 and 2 of Table 4

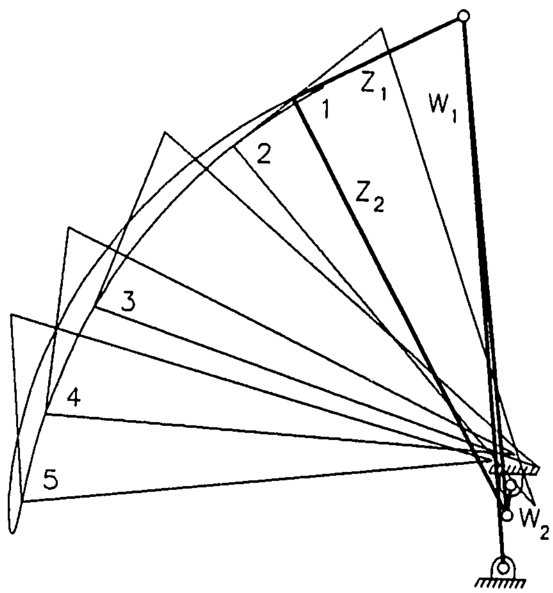

Figure 7: A mechanism combining dyads 1 and 3 of Table 4

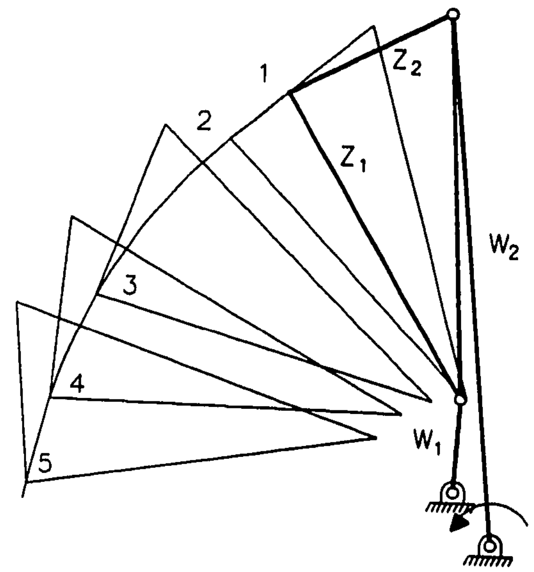

Figure 8: A mechanism combining dyads 1 and 4 of Table 4

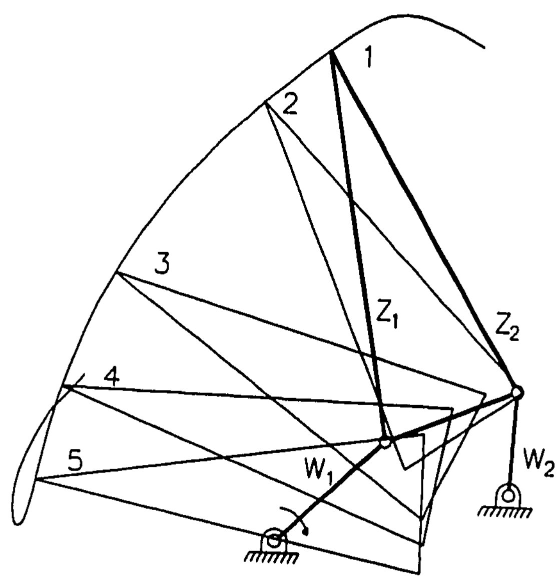

Figure 9: A mechanism combining dyads 2 and 4 of Table 4

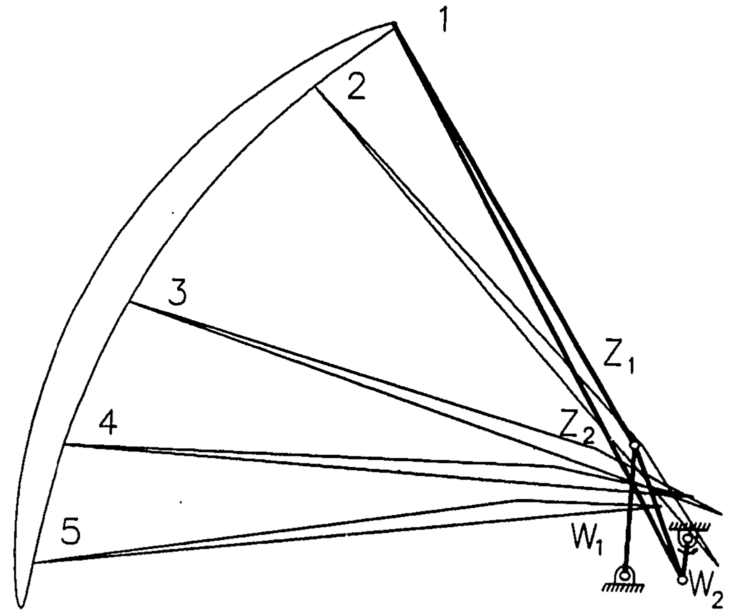

Figure 10: A mechanism combining dyads 3 and 4 of Table 4

REFERENCES

Chase, T. R., A. G. Erdman, and D. R. Riley. 1985. Improved Centerpoint Curve Generation Techniques for Four-Precision Position Synthesis using the Complex Number Approach. Journal of Mechanisms, Transmissions, and Automation in Design 107:370-376.

Erdman, A. G. 1981. Three and Four Precision Point Kinematic Synthesis of Planar Linkages. Journal of Mechanism and Machine Theory 16:227-245.

Morgan, A. 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. GM Research Publication, GMR 3651.

Morgan, Alexander. 1987. Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Roth, B., and F. Freudenstein. 1963. Synthesis of Path Generating Mechanisms by Numerical Methods. Journal of Engineering for Industry 85B, no.3:298-306.

Sandor, G. N., and A. G. Erdman. 1984. Advanced Mechanism Design: Analysis and Synthesis. Vol. 2. Prentice-Hall, Englewood Cliffs, New Jersey.

Subbian, T., and D. R. Flugrad. 1989. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation–1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:425-432.

PART II.

FOUR-BAR PATH GENERATION SYNTHESIS BY A CONTINUATION METHOD

INTRODUCTION

Synthesis of four-bar path generating mechanisms has been accomplished in the past by both graphical and analytical methods. Lindholm [1969] applied the point position reduction technique to synthesize five and six position path generation linkages. Suh and Radcliffe [1983] used Newton's method in arriving at a solution. An analytical approach developed by Sandor and Erdman [1984] successfully reduces the system of equations associated with the five position path generation problem with prescribed timing to a quartic polynomial equation which then can be solved.

Closed form solutions are effective up to four precision points and can be used as a tool to solve five specified points, but for six points and beyond the nonlinear equations are difficult to handle. Graphical methods have been applied for six precision points but with only limited success. Furthermore, the solution set obtained by graphical methods is not complete. This leads to the application of numerical methods that can theoretically be used to solve for a maximum of nine prescribed precision points. However, the practical limit has been five or six specified positions. On the other hand, if exact precision is not required, least square methods [Sarkisyan et al., 1973] and selective precision synthesis [Kramer and Sandor, 1975 and Kramer, 1979] have been used to minimize the deviation from a path described by more than nine points.

Numerical methods presently used for precision point synthesis of path generating mechanisms exhibit two major shortcomings:

- Convergence depends on reasonably good approximations for the solutions.

- Current numerical methods converge to a single solution dependent on the initial estimates, and thus do not yield all designs that satisfy the constraints.

The continuation method [Morgan, 1987] is a mathematical procedure that theoretically assures convergence without any initial solution estimates and that also produces the complete solution set for the given system of nonlinear equations, provided they are cast in polynomial form. The continuation method is used here as a tool to solve the five position synthesis problem. The procedure can effectively be extended to six or more precision points without difficulty. However, the computational expense increases significantly.

Similar methods were implemented by Freudenstein and Roth [1963] for synthesis of geared five-bar mechanisms, and by Tsai and Morgan [1985] for the analysis of five and six degree of freedom manipulators. Roth and Freudenstein [1963] made use of a variation of the method discussed here for the synthesis of nine position, path generating, geared five-bar and four-bar mechanisms. Nine points on the coupler path of an arbitrary mechanism were chosen as the starting points. A number of subproblems were solved to move from these points toward the nine prescribed positions, thereby developing the desired mechanism. The method to be used here starts with a simple set of equations involving the link lengths and moves toward the set of equations for the five prescribed precision points. In doing so, it moves from the starting solution of a simple mathematical system to the desired result. The method is set up to determine all possible solutions for the system of equations at hand while Roth and Freudenstein [1963] looked at one solution only.

The continuation method and its application to the synthesis of path generating mechanisms is described in detail in the following sections. Nonlinear loop closure equations are developed and modified for efficient implementation. The numerical method is then applied to determine the solutions for two five position synthesis problems.

CONTINUATION METHOD

Continuation methods constitute a family of mathematical procedures used to solve systems of nonlinear equations. These methods are particularly useful for dealing with sets of polynomial equations. To implement the approach, one starts with a system of equations for which the the solutions are known and then marches along a path toward the solutions of the original system. For simplicity, the procedure for a system of two equations in two unknowns is described here.

Consider two polynomial equations given by:

$$\begin{aligned} F_1(z_1, z_2) &= 0 \\ F_2(z_1, z_2) &= 0 \end{aligned} \tag{1}$$

To implement the method a simple system of two equations in two unknowns is first considered [Morgan, 1987].

$$\begin{aligned} G_1 &= C_{11} z_1^{d_1} - C_{12} = 0 \\ G_2 &= C_{21} z_2^{d_2} - C_{22} = 0 \end{aligned} \tag{2}$$

The terms C_{11}, C_{12}, C_{21} and C_{22} are randomly chosen complex constants, and d_1 and d_2 are the degree of functions F_1 and F_2, respectively. The necessary homotopy functions are then obtained by combining the two systems of Eqs. (1) and (2):

$$\begin{aligned} H_1(z_1, z_2, t) &= tF_1 + (1-t)G_1 = 0 \\ H_2(z_1, z_2, t) &= tF_2 + (1-t)G_2 = 0 \end{aligned} \tag{3}$$

Here t is called the homotopy parameter. When t=0, the homotopy functions reduce to the start set of equations and when t=1, they represent the target system. Therefore, by increasing t from 0 to 1, and by solving a number of intermediate subproblems along the way, the solutions for the original (target) system are found. There are a variety of ways to move from a t value of 0 to 1. In the present approach, basic differential equations (BDEs) [Garcia and Zangwill, 1981] are formed, and ordinary differential equations involving the variables with respect to t are determined. These differential equations are integrated numerically to determine the solution for the given system of equations. The solution is then refined using Newton's method. The procedure is repeated using all the various combinations of solutions for the assumed start system of equations as the initial values of the integration.

DEVELOPMENT OF EQUATIONS FOR PATH GENERATION SYNTHESIS

A four-bar mechanism in two finitely separated positions can be represented by the vectors shown in Fig. 1. Two loop closure equations are written, one for each dyad (vector pair). For vectors $\mathbf{Z_1}$ and $\mathbf{Z_2}$ we can establish that

$$\mathbf{Z_1}e^{i\phi_j} + \mathbf{Z_2}e^{i\gamma_j} = \delta_j + \mathbf{Z_1} + \mathbf{Z_2} \tag{4}$$

or

$$\mathbf{Z_1}(e^{i\phi_j} - 1) + \mathbf{Z_2}(e^{i\gamma_j} - 1) = \delta_j \tag{5}$$

where, j =2 to n, δ_j is the linear displacement of the coupler point from its initial position, ϕ_j is the angular displacement of $\mathbf{Z_1}$ vector from its initial configuration and γ_j is the change in the angular position of the coupler. Similarly, for vectors $\mathbf{Z_3}$ and $\mathbf{Z_4}$

$$\mathbf{Z_3}(e^{i\psi_j} - 1) + \mathbf{Z_4}(e^{i\gamma_j} - 1) = \delta_j \tag{6}$$

where ψ_j is the angular displacement of the vector $\mathbf{Z_3}$ from its initial position.

The $\mathbf{Z}$ vectors in Eqs. (5) and (6) can be represented in real and imaginary terms as $\mathbf{Z} = Z_x + iZ_y$. Upon substitution for $\mathbf{Z_1}$ through $\mathbf{Z_4}$, and for δ_j, in terms of real and imaginary components, and expansion of the exponentials using Euler's equation, Eqs. (5) and (6) may be written as

$$(Z_{1x} + iZ_{1y})(cos\phi_j - 1 + isin\phi_j) + (Z_{2x} + iZ_{2y})(cos\gamma_j - 1 + isin\gamma_j) = \delta_{jx} + i\delta_{jy} \tag{7}$$

and

$$(Z_{3x} + iZ_{3y})(cos\psi_j - 1 + isin\psi_j) + (Z_{4x} + iZ_{4y})(cos\gamma_j - 1 + isin\gamma_j) =$$
$$\delta_{jx} + i\delta_{jy} \tag{8}$$

To transform these equations into polynomial form, $cos\phi_j$ and $sin\phi_j$ are treated as two independent variables, $C\phi_j$ and $S\phi_j$ respectively. Constraint equations $C\phi_j^2 + S\phi_j^2 = 1$, etc, for angles ϕ_j, ψ_j and γ_j are introduced as well. Upon substitution of these constraints into Eq. (7) and separation into real and imaginary components, the following expressions result:

$$\begin{aligned} Z_{1x}C\phi_j - Z_{1x} - Z_{1y}S\phi_j + Z_{2x}C\gamma_j - Z_{2x} - Z_{2y}S\gamma_j &= \delta_{jx} \\ Z_{1x}S\phi_j + Z_{1y}C\phi_j - Z_{1y} + Z_{2x}S\gamma_j + Z_{2y}C\gamma_j - Z_{2y} &= \delta_{jy} \end{aligned} \tag{9}$$

Similarly for Eq. (8) we obtain

$$\begin{aligned} Z_{3x}C\psi_j - Z_{3x} - Z_{3y}S\psi_j + Z_{4x}C\gamma_j - Z_{4x} - Z_{4y}S\gamma_j &= \delta_{jx} \\ Z_{3x}S\psi_j + Z_{3y}C\psi_j - Z_{3y} + Z_{4x}S\gamma_j + Z_{4y}C\gamma_j - Z_{4y} &= \delta_{jy} \end{aligned} \tag{10}$$

Next, multiplication of the first of Eqs. (9) by Z_{1y}, and the second by Z_{1x}, followed by subtraction of the two eliminates the $C\phi_j$ terms:

$$\begin{aligned} S\phi_j = {} & (Z_{1x}\delta_{jy} - Z_{1y}\delta_{jx} - Z_{1x}Z_{2x}S\gamma_j + Z_{1y}Z_{2x}C\gamma_j - Z_{1x}Z_{2y}C\gamma_j \\ & -Z_{1y}Z_{2y}S\gamma_j + Z_{1x}Z_{2y} - Z_{1y}Z_{2x})/(Z_{1x}^2 + Z_{1y}^2) \end{aligned} \tag{11}$$

By a similar procedure, $S\phi_j$ is eliminated from Eqs. (9) to give

$$\begin{aligned} C\phi_j = {} & (Z_{1y}\delta_{jy} + Z_{1x}\delta_{jx} + (Z_{1x}^2 + Z_{1y}^2) - Z_{1y}Z_{2x}S\gamma_j - Z_{1x}Z_{2x}C\gamma_j \\ & -Z_{1y}Z_{2y}C\gamma_j + Z_{1x}Z_{2y}S\gamma_j + Z_{1y}Z_{2y} + Z_{1x}Z_{2x})/(Z_{1x}^2 + Z_{1y}^2) \end{aligned} \tag{12}$$

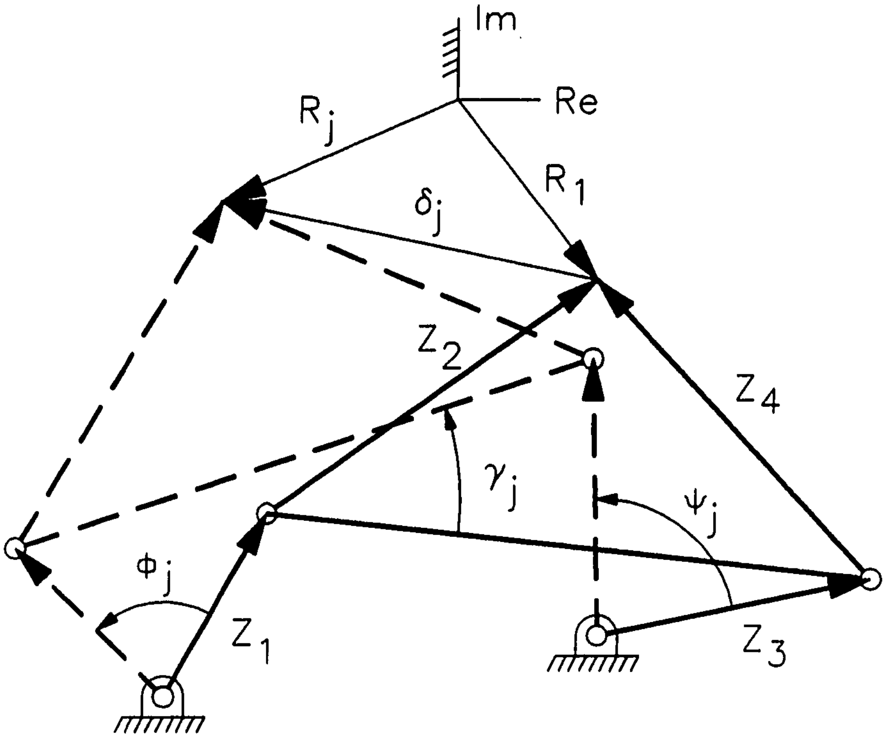

Figure 1: Four-Bar Mechanism in Two Finitely Separated Positions

Elimination of $S\phi_j$ and $C\phi_j$ between Eqs. (11) and (12), followed by rearrangement yields

$$\begin{aligned} &2(Z_{1x}Z_{2y} - Z_{1y}Z_{2x} + Z_{2y}\delta_{jx} - Z_{2x}\delta_{jy})S\gamma_j \\ &-2(Z_{2x}^2 + Z_{2y}^2 + Z_{1x}Z_{2x} + Z_{1y}Z_{2y} + Z_{2x}\delta_{jx} + Z_{2y}\delta_{jy})C\gamma_j \\ &= -2Z_{2x}^2 - 2Z_{2y}^2 - 2Z_{1x}Z_{2x} - 2Z_{1y}Z_{2y} - 2Z_{2x}\delta_{jx} \\ &-2Z_{2y}\delta_{jy} - 2Z_{1y}\delta_{jy} - 2Z_{1x}\delta_{jx} - \delta_{jx}^2 - \delta_{jy}^2 \end{aligned} \tag{13}$$

To simplify Eq. (13) further, the following quantities are defined.

$$\begin{aligned} A_{1j} &= Z_{1x}Z_{2y} - Z_{1y}Z_{2x} + Z_{2y}\delta_{jx} - Z_{2x}\delta_{jy} \\ B_{1j} &= Z_{2x}^2 + Z_{2y}^2 + Z_{1x}Z_{2x} + Z_{1y}Z_{2y} + Z_{2x}\delta_{jx} + Z_{2y}\delta_{jy} \\ D_{1j} &= 2Z_{1x}\delta_{jx} + 2Z_{1y}\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \end{aligned} \tag{14}$$

Thus Eq. (13) can be written as:

$$2A_{1j}S\gamma_j - 2B_{1j}C\gamma_j = -2B_{1j} - D_{1j} \tag{15}$$

A parallel development is pursued using Eqs. (10) to eliminate ψ_j. This yields

$$2A_{2j}S\gamma_j - 2B_{2j}C\gamma_j = -2B_{2j} - D_{2j} \tag{16}$$

where,

$$\begin{aligned} A_{2j} &= Z_{3x}Z_{4y} - Z_{3y}Z_{4x} + Z_{4y}\delta_{jx} - Z_{4x}\delta_{jy} \\ B_{2j} &= Z_{4x}^2 + Z_{4y}^2 + Z_{3x}Z_{4x} + Z_{3y}Z_{4y} + Z_{4x}\delta_{jx} + Z_{4y}\delta_{jy} \\ D_{2j} &= 2Z_{3x}\delta_{jx} + 2Z_{3y}\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \end{aligned} \tag{17}$$

Equations (15) and (16) are then solved for $S\gamma_j$ and $C\gamma_j$.

$$\begin{aligned} S\gamma_j &= (B_{1j}D_{2j} - B_{2j}D_{1j})/(2(A_{1j}B_{2j} - A_{2j}B_{1j})) \\ C\gamma_j &= 1 + (A_{1j}D_{2j} - A_{2j}D_{1j})/(2(A_{1j}B_{2j} - A_{2j}B_{1j})) \end{aligned} \tag{18}$$

Elimination of $S\gamma_j$ and $C\gamma_j$ then produces the final function, a polynomial in the complex variables $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$:

$$F_{j-1} = (B_{1j}D_{2j} - B_{2j}D_{1j})^2 + 4(A_{1j}B_{2j} - A_{2j}B_{1j})(A_{1j}D_{2j} - A_{2j}D_{1j})$$
$$+(A_{1j}D_{2j} - A_{2j}D_{1j})^2 = 0 \tag{19}$$

In Eq. (19) j =2 to n where n is the number of prescribed positions. Therefore, a five position path synthesis problem yields four equations. Since four complex $\mathbf{Z}$ variables are involved, the maximum number of positions is limited to nine. In the sections to follow Eq. (19) is used in conjunction with the continuation method to solve two example problems for five position synthesis.

THE FIVE POSITION SYNTHESIS PROBLEM

Having developed the equations for the path generation problem, we need to implement the continuation method to solve them. For a five position synthesis problem, a set of four polynomial equations is to be solved. Since the number of variables is greater than the number of equations, we have the freedom of selecting values for four of the eight complex components. Here, the coupler link vectors, $\mathbf{Z_2}$ and $\mathbf{Z_4}$, are specified to be $e + if$ and $g + ih$ respectively, with particular numerical values assigned to e,f,g and h. From Eq. (19) the four equations to be solved reduce to the following form:

$$F_{j-1} = (B_{1j}D_{2j} - B_{2j}D_{1j})^2 + 4(A_{1j}B_{2j} - A_{2j}B_{1j})(A_{1j}D_{2j} - A_{2j}D_{1j}) \\ +(A_{1j}D_{2j} - A_{2j}D_{1j})^2 = 0 \qquad (20)$$

where j=2, 3, 4, 5 and

$$\begin{aligned}
A_{1j} &= fZ_{1x} - eZ_{1y} + f\delta_{jx} - e\delta_{jy} \\
B_{1j} &= eZ_{1x} + fZ_{1y} + e^2 + f^2 + e\delta_{jx} + f\delta_{jy} \\
D_{1j} &= 2\delta_{jx}Z_{1x} + 2\delta_{jy}Z_{1y} + \delta_{jx}^2 + \delta_{jy}^2 \\
A_{2j} &= hZ_{3x} - gZ_{3y} + h\delta_{jx} - g\delta_{jy} \\
B_{2j} &= gZ_{1x} + hZ_{1y} + g^2 + h^2 + g\delta_{jx} + h\delta_{jy} \\
D_{2j} &= 2\delta_{jx}Z_{3x} + 2\delta_{jy}Z_{3y} + \delta_{jx}^2 + \delta_{jy}^2
\end{aligned}$$

Upon substitution and simplification, it is apparent that the degree of each of the equations is 4.

If we rule out the possibility of an infinite number of finite solutions, as well as an infinite number of solutions at infinity, Bezout's theorem [Morgan, 1987] guarantees in this case that the total number of finite solutions and solutions at infinity is 256, counting multiplicities. This number of solutions is equal to the product of the degrees of the individual equations.

To implement the continuation method, it is necessary to start with a set of polynomial equations (one for each of the four unknowns Z_{1x},Z_{1y},Z_{3x} and Z_{3y}) which is easy to solve. The degree of these equations should be at least equal to the degree of the original system. The start equations considered for the five position synthesis problem are:

$$\begin{aligned}
G_1 = C_{11}Z_{1x}^4 - C_{12} &= 0 \\
G_2 = C_{21}Z_{1y}^4 - C_{22} &= 0 \\
G_3 = C_{31}Z_{3x}^4 - C_{32} &= 0 \\
G_4 = C_{41}Z_{3y}^4 - C_{42} &= 0
\end{aligned} \tag{21}$$

where the C_{j1}'s and C_{j2}'s are randomly chosen complex constants. Each equation results in four complex solutions as follows:

$$Z_{1x} = r_1^{1/4}[cos(2\pi k/4 + \alpha_1/4) + isin(2\pi k/4 + \alpha_1/4)] \tag{22}$$

where (r_1, α_1) are polar coordinates of the ratio C_{12}/C_{11} and $k = 0, 1, 2, 3$. Similarly, Z_{1y}, Z_{3x} and Z_{3y} can be determined. The solutions thus obtained are grouped to form 256 combinations satisfying the starting system of equations. These provide 256 starting points for the continuation procedure.

Homotopy functions are defined next using the original equations, the start equations and the homotopy parameter t as follows:

$$H_j = tF_j + (1-t)G_j = 0, \qquad j = 1,2,3,4 \tag{23}$$

When $t = 0$ the homotopy functions reduce to the start set of equations, and when $t = 1$ they represent the equations to be solved to obtain a four-bar linkage passing through the five prescribed positions. Hence, on varying the parameter t from 0 to 1, we move away from the start system to the target system of equations. To do this we track the paths of the unknowns as we increment t.

For path tracking, we determine the first order derivatives of the variables with respect to t using the BDEs discussed earlier. These differential equations are integrated, using the solutions of the start system as initial values.

In order to determine the first order differential equations (derivatives) we need to first evaluate the extended Jacobian matrix of the homotopy function as indicated below:

$$DH = \begin{bmatrix} \frac{\partial H_1}{\partial Z_{1x}} & \frac{\partial H_1}{\partial Z_{1y}} & \frac{\partial H_1}{\partial Z_{3x}} & \frac{\partial H_1}{\partial Z_{3y}} & \frac{\partial H_1}{\partial t} \\ \frac{\partial H_2}{\partial Z_{1x}} & \frac{\partial H_2}{\partial Z_{1y}} & \frac{\partial H_2}{\partial Z_{3x}} & \frac{\partial H_2}{\partial Z_{3y}} & \frac{\partial H_2}{\partial t} \\ \frac{\partial H_3}{\partial Z_{1x}} & \frac{\partial H_3}{\partial Z_{1y}} & \frac{\partial H_3}{\partial Z_{3x}} & \frac{\partial H_3}{\partial Z_{3y}} & \frac{\partial H_3}{\partial t} \\ \frac{\partial H_4}{\partial Z_{1x}} & \frac{\partial H_4}{\partial Z_{1y}} & \frac{\partial H_4}{\partial Z_{3x}} & \frac{\partial H_4}{\partial Z_{3y}} & \frac{\partial H_4}{\partial t} \end{bmatrix}$$

Use of this Jacobian matrix allows the ordinary differential equations to be written as, dZ_{1x}/dt, dZ_{1y}/dt, dZ_{3x}/dt and $dZ_{3y}/dt = ((-1)^{j+1} det(DH_{[j]}))/Den$ for $j = 1, 2, 3, 4$ respectively. The $det(DH_{[j]})$ represents the determinant of the Jacobian matrix, DH, with the jth column deleted, and Den is $(-1)^{n+2} det(DH_{[n+1]})$.

Thus by integrating the differential equations obtained, we can determine all the solutions (complex, real and those at infinity) for the original system of equations. Among these, the real solutions are the useful ones for five position synthesis.

Example 1

A four-bar mechanism was designed to pass through the five precision points listed in Table 1 by applying the continuation method. To synthesize the mechanism, the links $\mathbf{Z_2}$ and $\mathbf{Z_4}$ were chosen to be $1.1344 + i1.3975$ and $-1.7287 + i0.5016$ respectively, and the complex link vectors $\mathbf{Z_1}$ and $\mathbf{Z_3}$ were computed. A total of 256 real, complex and solutions at infinity were obtained from the 256 starting points. Of these only twenty-five solutions, which are listed in Table 2, were real. Thirty-three pairs of complex conjugate solutions were obtained, and the rest either diverged to infinity or resulted in very large link lengths that caused the procedure to be aborted. Of the twenty-five real solutions found, the first five exhibited no branch or order defects and so were useful solutions. Three more solutions came close to satisfying the prescribed conditions, and they are discussed here as well.

Table 1: Prescribed precision points for example 1

Precision Point	X	Y
1	0	0
2	-0.4535	-0.1739
3	-0.8385	-0.5228
4	-1.0840	-0.9358
5	-1.1794	-1.2957

Of the eight useful solutions only the first four satisfy Grashof's criteria. All eight are shown in Figs. 2 through 9. Among them, the first two solutions resulted in crank rocker mechanisms and the third was a double crank mechanism. Mechanism 4, a double rocker mechanism, has very large link lengths. However there are situations

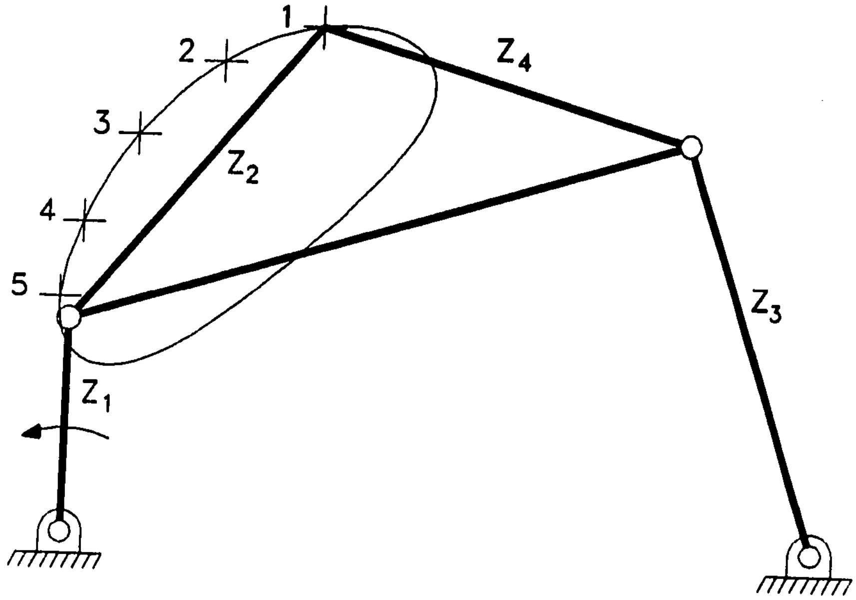

Figure 2: Mechanism 1 in its First Position With the Coupler Curve (Example 1)

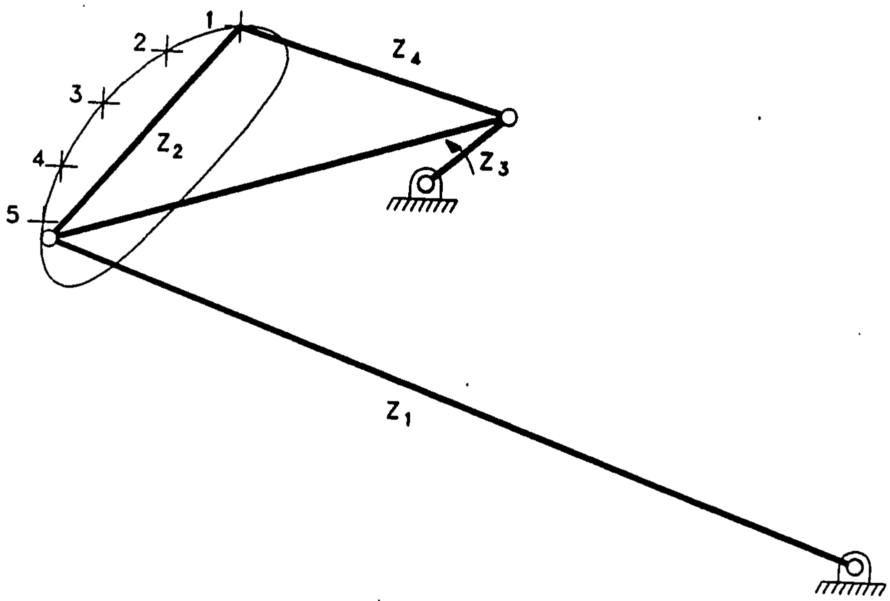

Figure 3: Mechanism 2 in its First Position With the Coupler Curve (Example 2)

in which such dimensions might be feasible, and so it is listed here as a valid solution. Solution 5 turned out to be a double rocker mechanism as well.

Mechanisms 6, 7 and 8 at the outset looked like multiple solutions (repeated solutions) but on further analysis were identified as being distinct and geometrically isolated. These solutions are discussed here despite the fact that they do not quite satisfy the specified order for the precision points. From Figs. 7, 8 and 9 it is clear that the coupler passes through the specified points in the order 1,5,4,3,2 and then comes close to passing through 1 again. If the device is driven backwards from the point where it is close to position 1, it will pass through 2,3,4,5 in the specified order. It should be noted, however, that when the coupler is at this new starting point the configuration is different from that shown in the figures. That is, the configuration is not consistent with the arbitrarily specified values for $\mathbf{Z_2}$ and $\mathbf{Z_4}$. Even so, these three linkages might still be usable to solve the path generation problem.

The above designs then provide us with eight four bar mechanisms for the task. Figs. 2 through 9 give the mechanisms in their initial position with the coupler curve superimposed. More mechanisms can be obtained, giving the designer more options, by changing the link vectors $\mathbf{Z_2}$ and $\mathbf{Z_4}$. For each of the link vectors chosen the procedure is carried out for the 256 starting points.

Example 2

The procedure was also applied to design a mechanism passing through five points lying on a straight line (Table 3). The links $\mathbf{Z_2}$ and $\mathbf{Z_4}$ were chosen to be $9.7226 + i2.1722$ and $2.5723 + i17.5314$ respectively. Twenty usable solutions were obtained with no order or branch defects. These are listed in Table 4. Of these

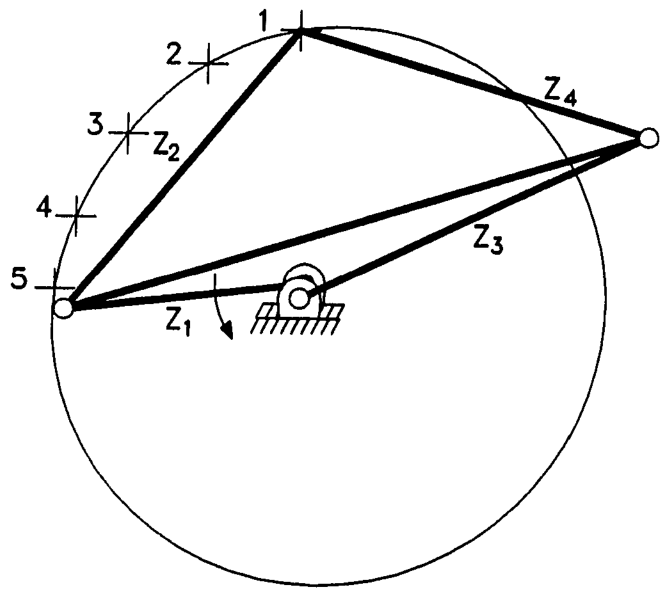

Figure 4: Mechanism 3 in its First Position With the Coupler Curve (Example 1)

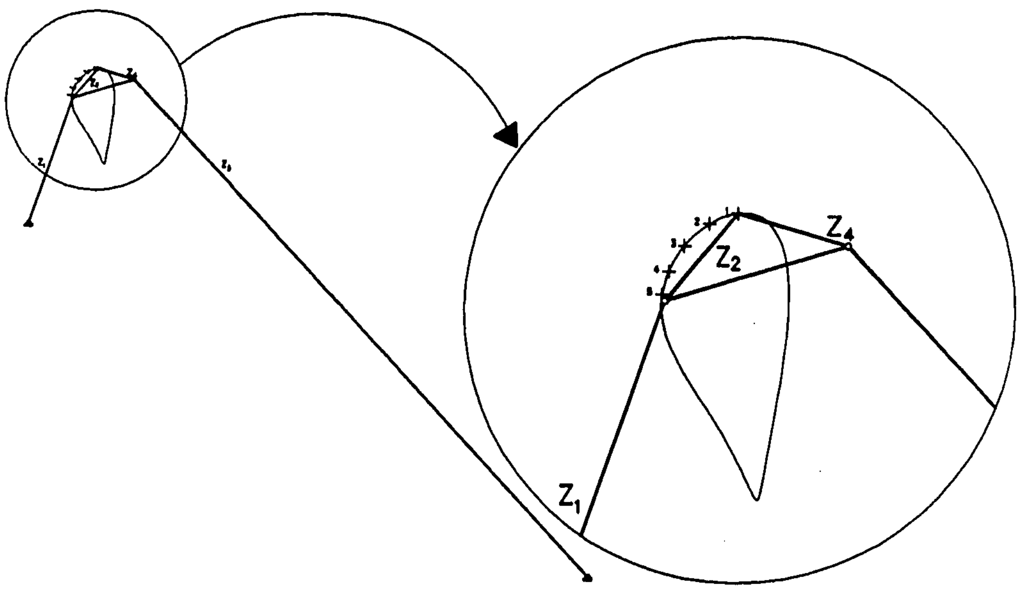

Figure 5: Mechanism 4 in its First Position With the Coupler Curve (Example 1)

Table 2: Continuation method solutions satisfying the prescribed conditions for example 1

No.	Z_{1x}	Z_{1y}	Z_{3x}	Z_{3y}	COMMENTS
1	0.0009	0.9997	-0.6386	1.8974	Grashof,crank rocker
2	-5.1960	1.8622	0.4997	0.4437	Grashof,crank rocker
3	-1.1852	-0.1413	1.6996	0.8254	Grashof,double crank
4	1.9835	5.9712	-21.9874	23.4323	Grashof,double rocker
5	0.2998	1.6070	-1.8717	3.0586	Non-Grashof,double rocker
6	-1.9668	-0.8894	2.0674	1.4717	Non-Grashof,double rocker
7	-1.8773	-0.9879	2.0039	1.6386	Non-Grashof,double rocker
8	-2.0930	-0.7404	2.1363	1.3139	Non-Grashof,double rocker
9	-0.9050	-0.2183	2.2164	0.7913	Grashof
10	-0.8583	0.2599	2.5453	2.5985	Grashof
11	0.6044	-0.8544	-1.4799	4.8112	Grashof
12	0.4295	1.4901	1.1392	-0.1488	Grashof
13	1.9769	-4.9389	2.1435	0.0899	Grashof
14	-2.2828	-0.5300	2.1898	1.1878	Grashof
15	-2.5162	-0.3240	2.2195	1.1085	Grashof
16	0.1328	-2.5004	2.1082	0.0314	Non-Grashof
17	0.1401	0.8555	0.8157	-0.2319	Non-Grashof
18	1.0298	-1.0562	3.5144	-3.7175	Non-Grashof
19	0.0467	0.9137	0.9355	-0.2420	Non-Grashof
20	-0.0567	1.0265	1.0408	-0.2781	Non-Grashof
21	-0.0121	0.9483	0.9808	-0.2618	Non-Grashof
22	0.3723	0.8183	0.6657	-0.2566	Non-Grashof
23	-0.5502	-1.5196	2.0264	0.0965	Non-Grashof
24	0.7951	-0.5396	-8.6096	12.1160	Non-Grashof
25	1.3553	-0.7603	4.3469	-4.6379	Non-Grashof

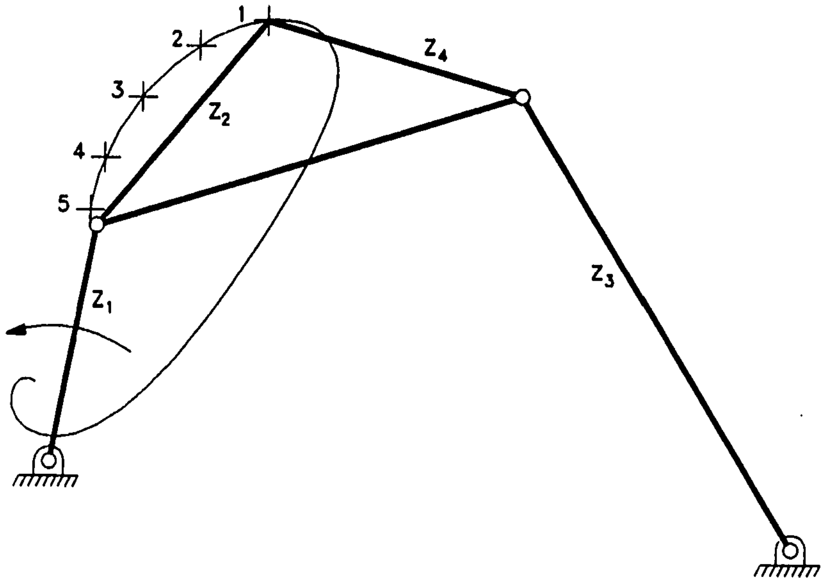

Figure 6: Mechanism 5 in its First Position With the Coupler Curve (Example 1)

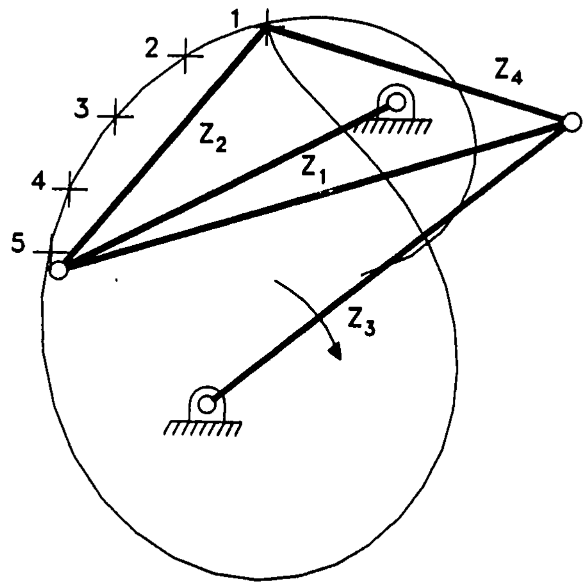

Figure 7: Mechanism 6 in its First Position With the Coupler Curve (Example 1)

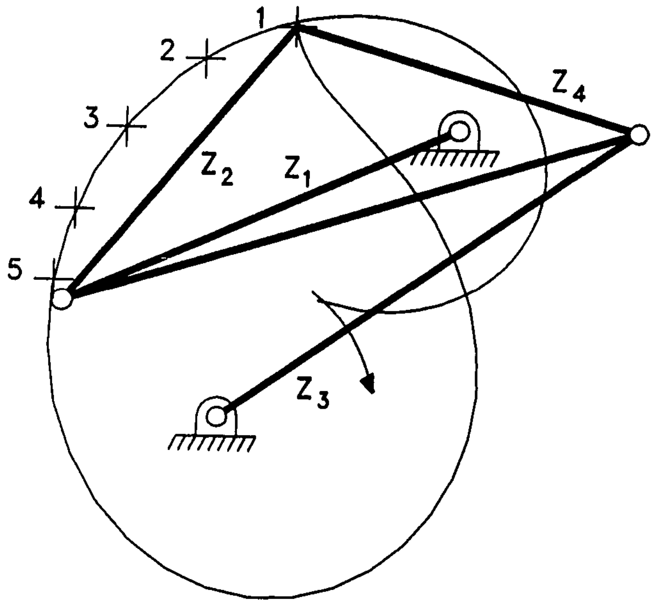

Figure 8: Mechanism 7 in its First Position With the Coupler Curve (Example 1)

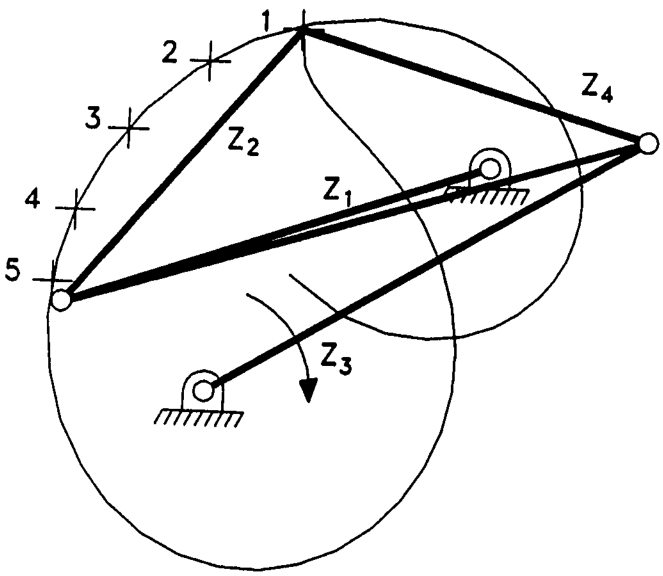

Figure 9: Mechanism 8 in its First Position With the Coupler Curve (Example 1)

Table 3: Prescribed precision points for example 2

Precision Point	X	Y
1	0	0
2	-3	0
3	-6	0
4	-9	0
5	-12	0

twenty, eight were Grashof double rocker mechanisms, and the rest were non-Grashof double rocker linkages. No usable crank rocker solutions were obtained. One of the Grashof double rocker mechanisms is shown in Fig. 10 with its coupler path and precision points. Fig. 11 depicts a non-Grashof double rocker mechanism. A couple of the solutions produced quite unexpected results. One of them is displayed in Fig. 12 which shows the coupler point passing through all five precision points in the proper order, but a full loop is executed between the third and fourth points. For that reason this particular design might not be acceptable.

Computations for the above examples were carried out in a VAX/VMS (11/785) environment. The average CPU time for each starting point was 26.65 seconds for the first example and 46.56 seconds for the second. When the solution diverged to infinity the CPU time was 60 seconds on average for example 1 and 100 seconds for example 2. For converging solution paths the CPU times were considerably less. The values were approximately 20 and 30 seconds for the first and second problems respectively.

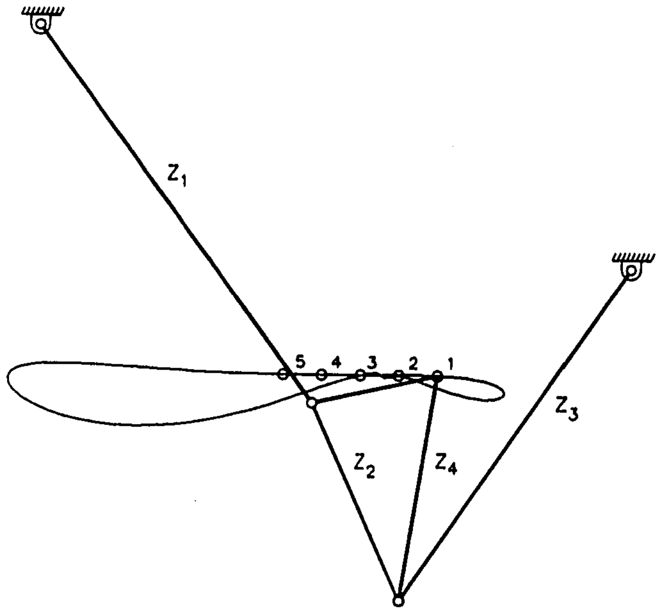

Figure 10: Mechanism 1 in its First Position With the Coupler Curve (Example 2)

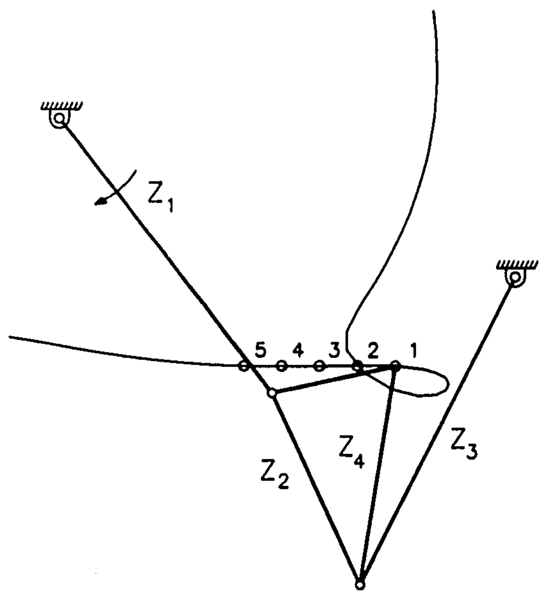

Figure 11: Mechanism 9 in its First Position With the Coupler Curve (Example 2)

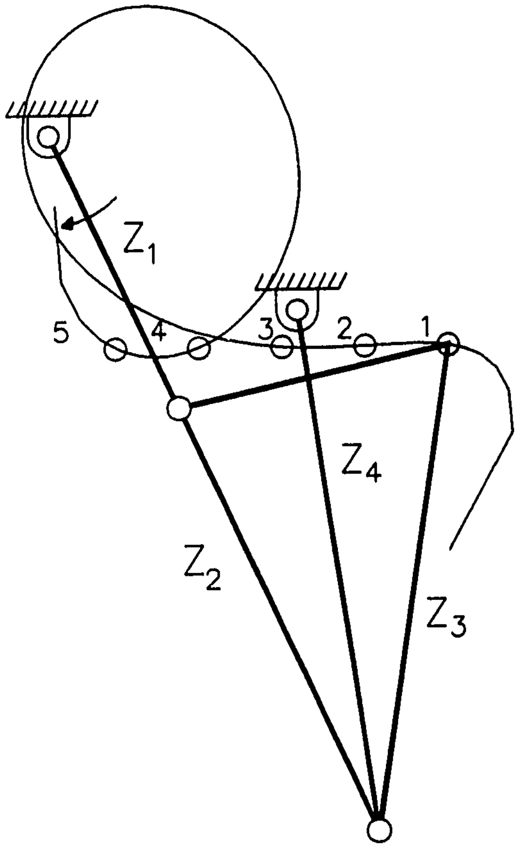

Figure 12: Mechanism 19 in its First Position With the Coupler Curve (Example 2)

Table 4: Continuation method solutions satisfying the prescribed conditions for example 2

No.	Z_{1x}	Z_{1y}	Z_{3x}	Z_{3y}	COMMENTS
1	21.725	-29.334	-17.410	-26.083	Grashof,double rocker
2	37.915	-53.216	-51.574	-34.365	Grashof,double rocker
3	21.806	-58.840	2.966	-17.521	Grashof,double rocker
4	39.436	-55.532	-49.675	-32.166	Grashof,double rocker
5	23.716	-64.374	2.479	-17.364	Grashof,double rocker
6	-44.669	50.802	55.339	-4.733	Grashof,double rocker
7	35.249	-49.166	-53.763	-38.150	Grashof,double rocker
8	42.292	-59.879	-41.478	-25.550	Grashof,double rocker
9	17.125	-21.851	-12.011	-24.651	Non-Grashof,double rocker
10	11.872	-12.500	-8.037	-25.530	Non-Grashof,double rocker
11	-0.016	4.074	2.128	-9.682	Non-Grashof,double rocker
12	-22.630	23.825	10.239	-16.170	Non-Grashof,double rocker
13	-3.394	3.400	-1.106	-9.701	Non-Grashof,double rocker
14	-0.184	4.108	3.298	-13.797	Non-Grashof,double rocker
15	-27.774	30.043	16.779	-14.634	Non-Grashof,double rocker
16	-9.345	-10.556	-2.861	-32.327	Non-Grashof,double rocker
17	1.433	3.784	1.503	-10.734	Non-Grashof,double rocker
18	12.722	-15.122	-7.535	-23.276	Non-Grashof,double rocker
19	4.627	-9.862	2.871	-18.844	Non-Grashof,double rocker
20	0.148	6.971	1.863	-14.603	Non-Grashof,double rocker

REFERENCES

Freudenstein, F. and B. Roth. 1963. Numerical Solution of Systems of Nonlinear Equations. Journal of the Association of Computing Machinery 10:550-556.

Garcia, C. B., and W. I. Zangwill. 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, New Jersey.

Kramer, S. N., and G. N. Sandor. 1975. Selective Precision Synthesis - A General Method of Optimization for Planar Mechanisms. Journal of Engineering for Industry 97B, no.2:689-701.

Kramer, S. N. 1979. Selective Precision Synthesis of the Four-Bar Motion Generator with Prescribed Input Timing. Journal of Mechanical Design 101:614-618.

Lindholm, J. C. 1969. A Survey of the Graphical Techniques in Designing for Specific Input-Output Relationships of a Four-Bar Mechanism. Proceedings of the First Applied Mechanisms Conference, Stillwater, Oklahoma (Oklahoma State University):35-1-35-6.

Morgan, A. 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. General Motors Research Laboratories, Warren, Michigan (GMR-3651).

Morgan, Alexander. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Roth, B., and F. Freudenstein. 1963. Synthesis of Path-Generating Mechanisms by Numerical Methods. Journal of Engineering for Industry 85B, no.3:298-306.

Sandor, G. N., and A. G. Erdman. 1984. Advanced Mechanism Design: Analysis and Synthesis; Vol. 2. Prentice-Hall, Englewood Cliffs, New Jersey.

Sarkisyan, Y. L., K. C. Gupta, and B. Roth. 1973. Kinematic Geometry Associated with the Least-Square Approximation of a Given Motion. Journal of Engineering for Industry 95, no.2:503-510.

Suh, C. H., and C. W. Radcliffe. 1983. Kinematics and Mechanisms Design. Robert E. Krieger Publishing Company, Malabar, Florida.

Tsai, L. W., and A. P. Morgan. 1985. Solving the Kinematics of the most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods. Journal of Mechanisms, Transmissions and Automation in Design 107:189-200.

PART III.

FIVE POSITION TRIAD SYNTHESIS WITH APPLICATIONS TO FOUR- AND SIX-BAR MECHANISMS

INTRODUCTION

Synthesis of four-bar function generation, five-bar and many multi-loop mechanisms can be accomplished by systematically breaking up the linkage into dyads and triads. The maximum number of precision points satisfied by a dyad is five for path generation with prescribed timing and for motion generation problems. This restricts the triad synthesis to five prescribed points whenever it is combined with dyads, and so this part confines itself to five position triad synthesis.

Synthesis of dyads to solve motion generation and path generation with prescribed timing has already been achieved [Erdman, 1981, Subbian and Flugrad, 1989a]. Furthermore, triad synthesis for motion generation with prescribed timing has been of interest lately. Chase et al. [1987] simplified the triad synthesis problem to one of dyad synthesis and carried out the design for up to five precision points. Since they transformed a triad to a dyad, the maximum number of precision points was limited to five for which a total of up to four useful solutions can be obtained. This work differs from Chase et al. [1987] in the methodology used and in the fact that a family of solution curves is generated as opposed to a finite number of solutions.

Lin and Erdman [1987] used the compatibility linkage approach, similar to the formulation carried out by Chase et al. [1985] for dyads to obtain the triad Burmester curves for six precision points. Subbian and Flugrad [1990] have achieved the triad synthesis for both six and seven prescribed precision points using continuation methods. The procedure used is similar to the one presented here.

The continuation method used for designing the triad in the current approach is a mathematical procedure to solve systems of n polynomial equations in n or $(n+1)$ unknowns. The five position motion generation with prescribed timing problem under consideration results in four equations involving six unknowns. This means there are two infinities of triads which satisfy the prescribed conditions. An attempt is made to generate a set of design curves which will enable the designer to pick a mechanism best suited for the application under consideration.

The applicability of continuation methods for kinematic synthesis problems has been established through the work of Subbian and Flugrad [1989a, 1989b], Morgan and Wampler [1989], Wampler et al. [1990], and Tsai and Lu [1989]. A detailed description of the procedure can be obtained from the above references or the book by Morgan [1987].

In sections to follow, equations for the synthesis of a triad are developed, and the solution procedure for a five position motion generation with prescribed timing problem outlined. The method is then applied to design a four-bar function generation mechanism and a Stephenson III type six-bar mechanism.

APPLICATIONS

Triads are useful in the design of four-bar, five-bar and a variety of multi-loop mechanisms. Fig. 1 shows a four-bar function generation mechanism where the relationship between the input angle, ψ_j, and the output angle, γ_j, is specified and the link lengths are to be determined. The vectors $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ form a triad with the fixed pivot at the base of $\mathbf{Z}_1$ and the coupler point at the base of $\mathbf{Z}_3$. On using the triad synthesis procedure and specifying the coupler displacements to be zero, we can design a four-bar function generation mechanism.

The five-bar mechanism (Fig. 2) can be designed by reducing it to a combination of a dyad and a triad. Links $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ constitute the triad, and $\mathbf{Z}_4$, $\mathbf{Z}_5$ form the dyad. The dyad and triad are designed independently and then connected to obtain the desired five-bar mechanism.

The Stephenson III type six-bar mechanism and the Watt II type mechanism are shown in Fig. 3. The Stephenson III mechanism consists of the triad $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and a pair of dyads $\mathbf{Z}_4$, $\mathbf{Z}_5$ and $\mathbf{Z}_6$, $\mathbf{Z}_7$. The Watt II mechanism can be designed by the synthesis of the four-bar motion generating mechanism ($\mathbf{Z}_4$, $\mathbf{Z}_5$, $\mathbf{Z}_6$ and $\mathbf{Z}_7$). From that we obtain the γ_j's corresponding to the prescribed precision points. With those and the specified ψ_j values, the triad $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ is designed in a manner similar to the four bar function generating mechanism.

Thus, most complex mechanisms can be broken into components consisting of dyads and triads. Dyad synthesis, as mentioned previously, has been addressed by a number of authors, and triad synthesis is considered here.

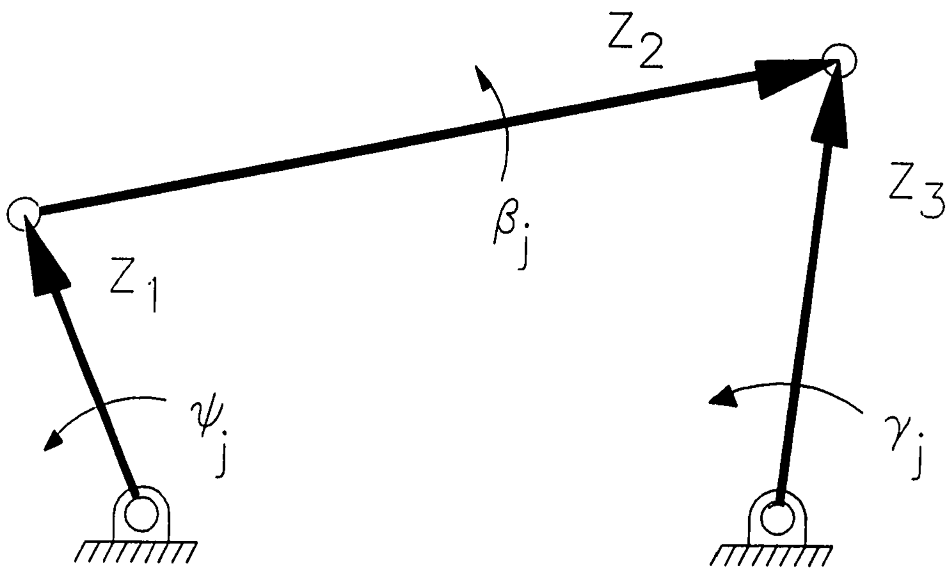

Figure 1: Four-bar function generating mechanism

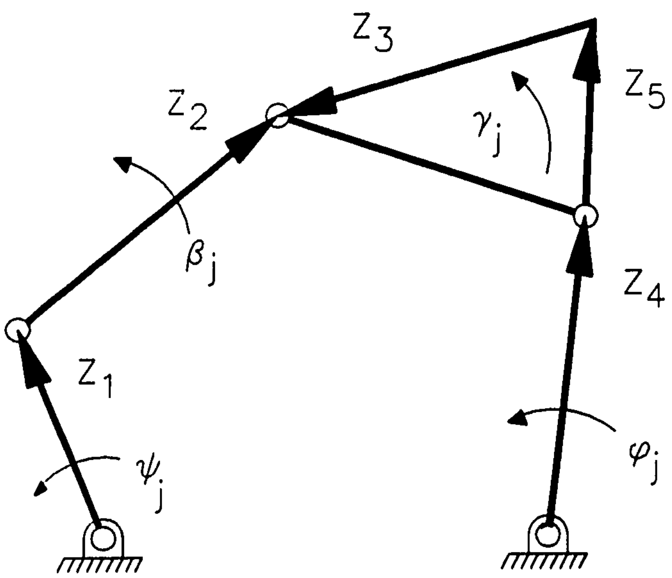

Figure 2: Five-bar mechanism

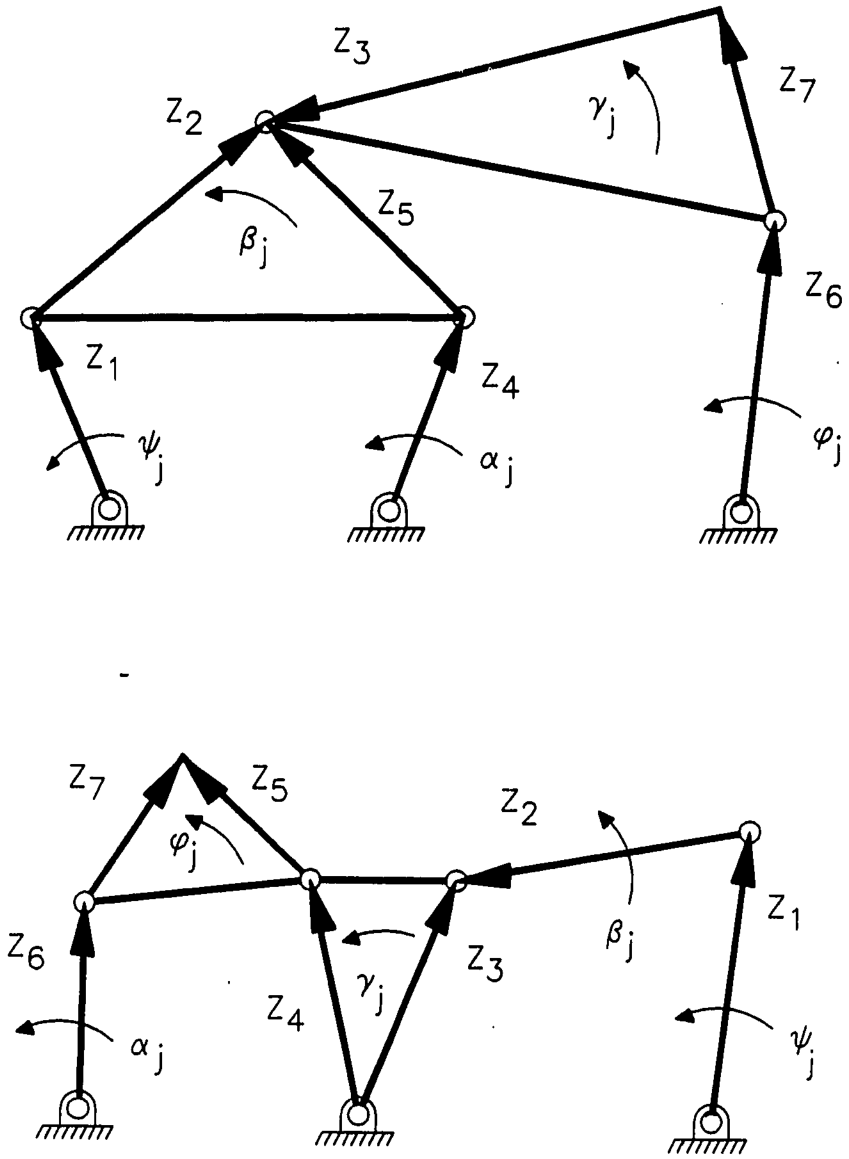

Figure 3: Stephenson III and Watt II six-bar mechanisms

DISPLACEMENT EQUATIONS FOR A TRIAD

A triad in two finitely separated positions is shown in Fig. 4. The angular displacements ψ_j and γ_j are specified along with the coupler displacements, δ_j's. The loop closure equation for the triad can be written, using complex numbers, as follows:

$$\mathbf{Z_1}(e^{i\psi_j} - 1) + \mathbf{Z_2}(e^{i\beta_j} - 1) - \mathbf{Z_3}(e^{i\gamma_j} - 1) = \delta_j \tag{1}$$

Eq. (1) is expressed as two independent equations by separating the real and imaginary terms. In doing so we end up with the following:

$$\begin{aligned}
Z_{1x}(cos\psi_j - 1) \; &- \; Z_{1y}sin\psi_j + Z_{2x}(cos\beta_j - 1) - Z_{2y}sin\beta_j \\
&- \; Z_{3x}(cos\gamma_j - 1) + Z_{3y}sin\gamma_j = \delta_{jx} \\
Z_{1y}(cos\psi_j - 1) \; &+ \; Z_{1x}sin\psi_j + Z_{2y}(cos\beta_j - 1) + Z_{2x}sin\beta_j \\
&- \; Z_{3y}(cos\gamma_j - 1) - Z_{3x}sin\gamma_j = \delta_{jy}
\end{aligned} \tag{2}$$

By treating $cos\beta_j$ and $sin\beta_j$ as two independent variables, $C\beta_j$ and $S\beta_j$ respectively, and by introducing the constraint relationship $(C\beta_j^2 + S\beta_j^2 = 1)$, we can reduce the above equations to polynomial form with ψ_j and γ_j values specified:

$$F_{j-1} \; = \; 2A_j sin\psi_j - 2B_j(cos\psi_j - 1) + D_j \tag{3}$$

where $j = 2$ to n, and

$$\begin{aligned}
A_j \; &= \; Z_{2x}Z_{1y} - Z_{2y}Z_{1x} + Z_{1y}\delta_{jx} - Z_{1x}\delta_{jy} + (cos\gamma_j - 1)[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}] \\
&- \; sin\gamma_j[Z_{1y}Z_{3y} + Z_{1x}Z_{3x}] \\
B_j \; &= \; Z_{1x}^2 + Z_{1y}^2 + Z_{2x}Z_{1x} + Z_{2y}Z_{1y} + Z_{1x}\delta_{jx} + Z_{1y}\delta_{jy} \\
&+ \; (cos\gamma_j - 1)[Z_{3x}Z_{1x} + Z_{3y}Z_{1y}] + sin\gamma_j[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}]
\end{aligned}$$

$$\begin{aligned}
D_j &= 2Z_{2x}\delta_{jx} + 2Z_{2y}\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \\
&+ 2(cos\gamma_j - 1)[Z_{3x}Z_{2x} + Z_{3y}Z_{2y} + Z_{3x}\delta_{jx} + Z_{3y}\delta_{jy} - Z_{3x}^2 - Z_{3y}^2] \\
&+ 2\sin\gamma_j[Z_{3x}Z_{2y} - Z_{3y}Z_{2x} + Z_{3x}\delta_{jy} - Z_{3y}\delta_{jx}]
\end{aligned}$$

Replacement of the variables Z_{2x} and Z_{2y} by polar R and θ variables is helpful for obtaining the solution curves. The newly defined variables correspond to the distance and orientation of the vector connecting the fixed pivot of the triad and the coupler point in its initial configuration, as indicated in Fig. 5. The relationship between these variables and the link vectors can be written as:

$$\begin{aligned}
Z_{2x} &= Rcos\theta + Z_{3x} - Z_{1x} \\
Z_{2y} &= Rsin\theta + Z_{3y} - Z_{1y}
\end{aligned} \tag{4}$$

Substituting for Z_{2x} and Z_{2y} in Eq. (3) and rearranging the terms, we obtain:

$$\begin{aligned}
F_{j-1} &= A_{1j}(Z_{3x}Z_{1x} + Z_{3y}Z_{1y}) + A_{2j}(Z_{3x}Z_{1y} - Z_{3y}Z_{1x}) + A_{3j}Z_{3x} \\
&+ A_{4j}Z_{3y}A_{5j}Z_{1x} + A_{6j}Z_{1y} + A_{7j}
\end{aligned} \tag{5}$$

where $j = 2$ to n, and

$$\begin{aligned}
A_{1j} &= -2sin\psi_j sin\gamma_j - 2cos\psi_j cos\gamma_j + 2 \\
A_{2j} &= 2sin\psi_j cos\gamma_j - 2cos\psi_j sin\gamma_j \\
A_{3j} &= 2(cos\gamma_j - 1)Rcos\theta + 2cos\gamma_j\delta_{jx} + 2sin\gamma_j\delta_{jy} + 2sin\gamma_j Rsin\theta \\
A_{4j} &= 2(cos\gamma_j - 1)Rsin\theta + 2cos\gamma_j\delta_{jy} - 2sin\gamma_j\delta_{jx} - 2sin\gamma_j Rcos\theta \\
A_{5j} &= -2sin\psi_j(Rsin\theta + \delta_{jy}) - 2(cos\psi_j - 1)(Rcos\theta + \delta_{jx}) - 2\delta_{jx} \\
A_{6j} &= 2sin\psi_j(Rcos\theta + \delta_{jx}) - 2(cos\psi_j - 1)(Rsin\theta + \delta_{jy}) - 2\delta_{jy} \\
A_{7j} &= 2Rcos\theta\delta_{jx} + 2Rsin\theta\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2
\end{aligned}$$

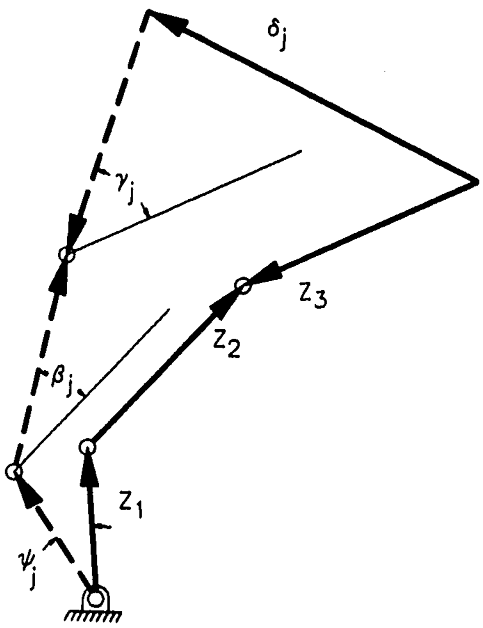

Figure 4: A triad in two finitely separated positions

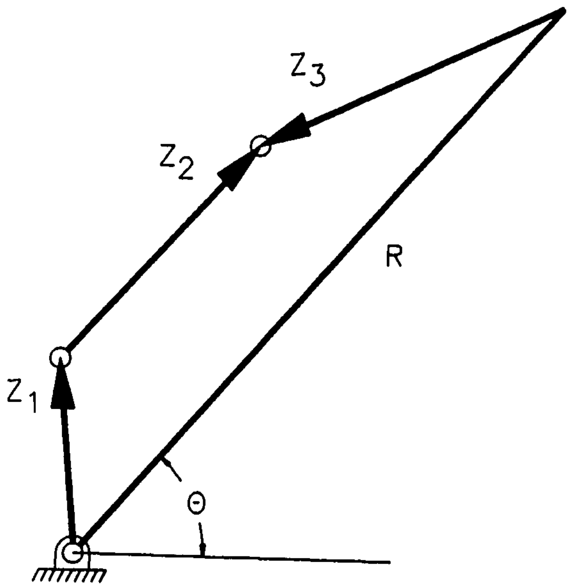

Figure 5: Triad showing the newly defined variables (R and θ)

DISPLACEMENT EQUATIONS FOR A DYAD

Fig. 6 shows a dyad in two finitely separated positions. The kinematic equations from the loop closure equation are expressed as [Subbian and Flugrad, 1989a]:

$$E_{j-1} = 2A_j sin\alpha_j + 2B_j(1 - cos\alpha_j) + D_j \tag{6}$$

where j = 2 to n, and

$$\begin{aligned} A_j &= W_xV_y - W_yV_x + V_y\delta_{jx} - V_x\delta_{jy} \\ B_j &= V_x^2 + V_y^2 + W_xV_x + W_yV_y + V_x\delta_{jx} + V_y\delta_{jy} \\ D_j &= 2W_x\delta_{jx} + 2W_y\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \end{aligned}$$

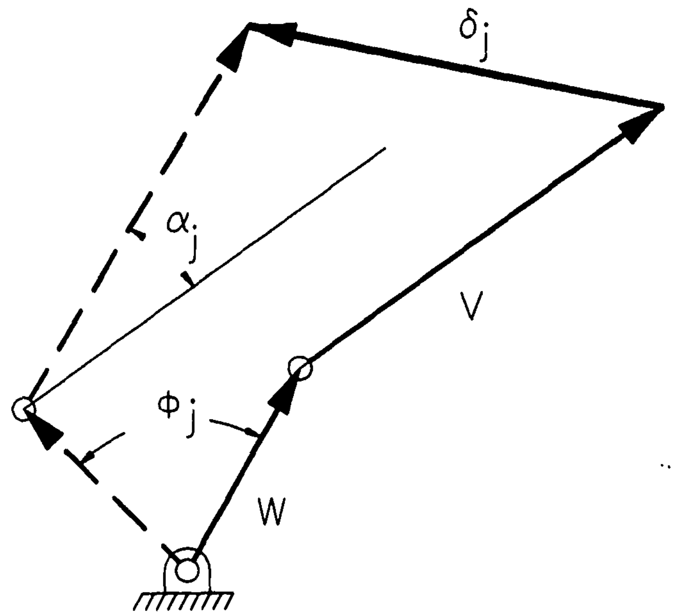

Figure 6: A dyad in two finitely separated positions

FIVE POSITION TRIAD SYNTHESIS USING CONTINUATION METHODS

Five position motion generation with prescribed timing problems result in four second degree polynomial equations in six unknowns. These equations are obtained for $j = 2, 3, 4$ and 5 in Eq. (5). The unknowns are then Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y}, R and θ, when the linear and angular displacements, δ_j, ψ_j, γ_j, are specified. Since the number of unknowns exceeds the number of equations by two, the system has two infinities of solutions.

On fixing R and θ a system of four equations in four unknowns is obtained. Solution of these four equations will result in points on the solution curves. R and θ were selected as the free choices to enable the designer to position the fixed pivot at a convenient location with respect to the coupler point.

Once specific points on the solution curves are obtained, the curves are traced by freeing the variables R and θ, one at a time, and by using a continuation method to solve the set of four equations in five unknowns. The methods used to solve the five position triad synthesis problem are outlined in the sections to follow.

To solve 4 equations in 4 unknowns

The system under consideration can be solved using a traditional 1-homogeneous homotopy, or a modified and more efficient homotopy. Since we are dealing with four polynomial equations, each of degree two, a total of 16 (2^4) paths must be tracked to obtain all the solutions when using a 1-homogeneous homotopy. In order to obtain a computationally efficient homotopy, the triad equations were solved for randomly chosen linear and angular displacements using the 1-homogeneous homotopy. Twelve of the sixteen paths tracked resulted in solutions at infinity, and four complex solutions in conjugate pairs were obtained.

Upon expressing Eqs. (5) in 2-homogeneous form (refer to Appendix A) by grouping Z_{1x} and Z_{1y} as one set of variables and Z_{3x} and $Z_{3}y$ as the other, we can show that the Bezout number for the system is 6. This eliminates 10 of the 16 paths. A projective transformation was incorporated (refer to Appendix A), and the system of equations was solved for the arbitrary set of precision points. Transforming the solutions back to the original system, four finite solutions and two solutions at infinity were obtained.

The 1-homogeneous and 2-homogeneous homotopies indicate that the number of solutions of finite magnitude is 4 for the five position triad synthesis problem. Of these four solutions, depending on the prescribed precision points and the free choices, the number of real solutions will be zero, two or four. The procedure is set up using a secant homotopy requiring only four starting points. The homotopy function has the following form:

$$\mathbf{H}(y,t) = (1-t)ae^{i\sigma}\mathbf{G}(y) + t\mathbf{F}(y) \tag{7}$$

where, **H** represents the n homotopy functions

y is the n vector of variables

t is the homotopy parameter

σ is a randomly chosen angle

a is a randomly chosen constant

F is the system of n equations to be solved

G is the start system of n equations similar to **F**

The formulation used here along with other formulations are discussed in detail by Wampler et al. [1990] (refer to Appendix B).

The system solved using the 2-homogeneous homotopy was used as the start system, $\mathbf{G}(y) = 0$, for future synthesis problems. It was observed through a number of examples that the two solutions at infinity for the start system always resulted in solutions at infinity for the target system, $\mathbf{F}(y) = 0$. Thus these two paths could be dropped. This formulation decreased the CPU time drastically because of the reduction of the number of paths from 16 to 4, and because the start system closely resembled the system of equations to be solved.

The solution procedure is not complete without a path tracking scheme. An approach utilizing the so-called basic differential equations (BDEs) was used for path tracking [Subbian and Flugrad, 1989a, 1989b, and Garcia and Zangwill, 1981] as the homotopy parameter, t, was varied from 0 to 1. The differential equations used for path tracking are obtained as follows: dZ_{1x}/dt, dZ_{1y}/dt, dZ_{3x}/dt and $dZ_{3y}/dt = ((-1)^{j+1} det(DH_{[j]}))/Den$ for $j = 1, 2, 3, 4$, respectively. The $det(DH_{[j]})$ represents the determinant of the extended Jacobian matrix, DH, with the jth column deleted, and Den is the determinant with the fifth column deleted.

By integrating these differential equations between the limits of 0 and 1, with the four solutions of the start system as initial conditions, we move from the solutions of the start system toward the solutions of the system of equations under consideration. A variable step Adam's method was used to carry out the integration with an error tolerance of 10^{-5}, and the solutions were refined at $t = 1$ using Newton's method. Of the solutions obtained, only the real solutions were useful for triad synthesis. These real solutions were used to trace the solution curves described in the following section.

To solve 4 equations in 5 unknowns

Once discrete points on the solution curves are obtained using an arbitrary value for R and θ, the curves are generated by allowing R and θ to vary. R and θ are used instead of Z_{2x} and Z_{2y} so that the designer can more conveniently position the fixed pivot with respect to the coupler point location in the initial configuration. The upper and lower limits for R are selected to place the fixed pivots at reasonable distances from the coupler point. The angle θ is varied between 0 and 2π.

The procedure to solve a system of n equations in $(n+1)$ unknowns is discussed by Subbian and Flugrad [1989a] and Morgan [1981]. The first step is to obtain the extended Jacobian matrix which, in the present case, is made up of the partial derivatives of the functions to be solved (Eq. (5)) with respect to Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y} and either R or θ (depending on the variable to be varied). This matrix is used to obtain the differential equations of the variables with respect to R or θ. The differential equations thus generated are solved to obtain the required design curves.

First R is varied and the differential equations are integrated with the real solutions from solving 4 equations in 4 unknowns as initial conditions. R was varied between permissible, R_{min} and R_{max} values, and the corresponding Z_{1x}, Z_{1y}, Z_{3x} and Z_{3y} values determined for fixed θ. Next, specific values of R were selected between the permissible limits. For these R values, the calculated Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y} values were used as initial conditions, and the solution curves were generated allowing θ to vary. This integration would generate infinities of feasible triads.

In tracking solution curves using real solutions as initial conditions, the Jacobian matrix could become singular whenever the solutions go complex. Although this problem was a possibility it was not encountered in the examples presented here.

EXAMPLES

Synthesis of a four-bar function generator

A four-bar function generating mechanism, shown in Fig. 1, was designed to satisfy the relationship $y = 2x^2 - x$ for $0 \leq x \leq 2$. The range of input angle, $\Delta\psi$, was specified to be 45^o, and the range of the output angle, $\Delta\gamma$, to be 90^o. Chebyshev spacing was used to obtain the precision points listed in Table 1. The displacement of the coupler point was set to zero to carry out function generator design.

Table 1: Precision points for the design of the four-bar function generator

Precision Point	ψ_j deg.	γ_j deg.
2	8.173602	-0.423306
3	21.398769	15.662287
4	34.623938	52.477363
5	43.898771	85.595082

The free choices, R and θ, were chosen to be 3.0 and 0^o respectively, and the continuation method was used to solve the four resulting equations in four unknowns. Four real solutions were obtained as listed in Table 2. Of these, solution 4 is trivial. Fig. 7 shows mechanism 1 of Table 2 in the five prescribed positions.

Changing the free choice R scales the link lengths up or down, proportional to the change in R. Varying θ rotates the mechanism in the x-y plane without altering the link lengths. Therefore, the mechanism could be designed for any choice of R and θ by appropriately scaling and rotating the linkages listed in Table 2.

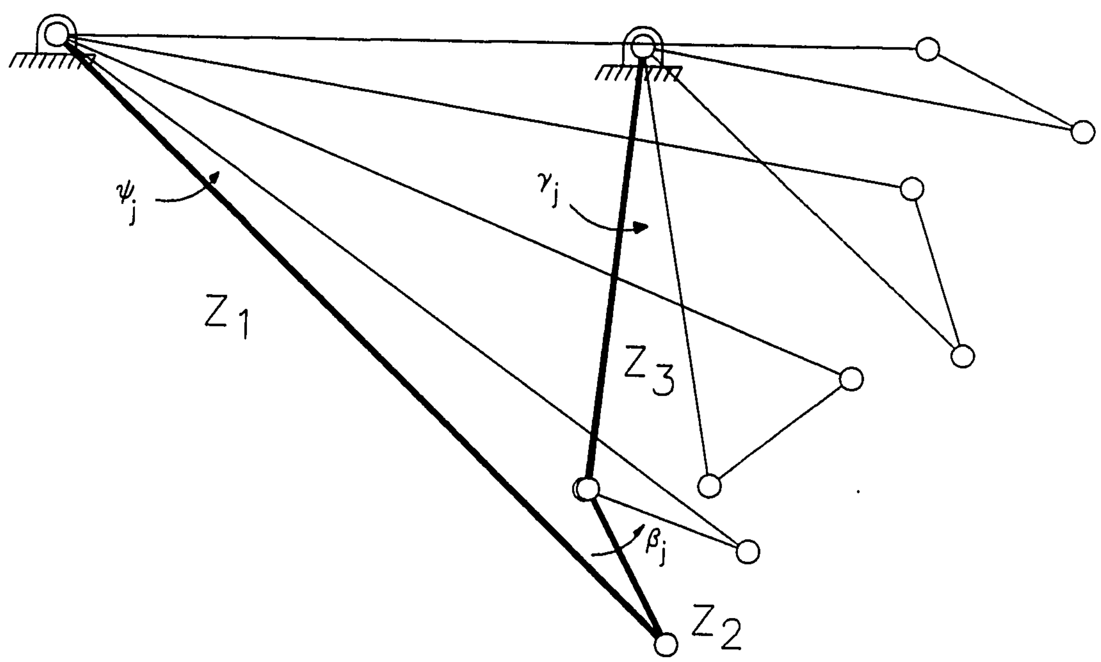

Figure 7: Triad 1 of Table 2 in the five prescribed positions

Table 2: Four-bar mechanisms satisfying the prescribed conditions (free choice $R = 3.0$ and $\theta = 0^o$)

No.	Z_{1x}	Z_{1y}	Z_{3x}	Z_{3y}
1	3.217734	-3.068392	-0.209436	-2.274785
2	4.735128	-2.103697	-0.279835	-0.958471
3	9.336607	-2.764468	-0.535495	-0.519322
4	0.0	0.0	0.0	0.0

Synthesis of a six-bar motion generating mechanism with prescribed timing

A Stephenson III type mechanism, shown in Fig. 3a, was designed for the precision points listed in Table 3. The mechanism was separated into a triad ($\mathbf{Z}_1$,$\mathbf{Z}_2$,$\mathbf{Z}_3$) designed for motion generation with prescribed timing, and two dyads ($\mathbf{Z}_4$,$\mathbf{Z}_5$ and $\mathbf{Z}_6$, $\mathbf{Z}_7$) satisfying the motion generation specifications.

Table 3: Precision points for the design of the six-bar mechanism

Precision Point	δ_{jx}	δ_{jy}	ψ_j deg.	γ_j deg.
2	-0.6331	-0.5449	20.0	12.65
3	-2.0713	-2.3566	40.0	42.65
4	-2.5510	-3.5456	60.0	57.65
5	-2.7720	-4.5210	80.0	67.65

In synthesizing the triad, the free choices R and θ were assumed to be 4.0 and 60^o, respectively. The four real solutions obtained are listed in Table 4. Next, the dyad $\mathbf{Z}_6$,$\mathbf{Z}_7$ was synthesized for the displacements and coupler orientations specified in Table 3. Again, four feasible solutions were obtained, as listed in Table 5. Before designing the $\mathbf{Z}_4,\mathbf{Z}_5$ dyad it was required to determine the precision points by analyzing the triad selected from Table 4. From the analysis of the triad, when the triad takes on the five prescribed positions, the angular displacements, β_j's, of the intermediate link of the triad, and the linear displacements δ'_j's of the tip of link $\mathbf{Z}_2$ were required.

Solving for $C\beta_j$ and $S\beta_j$ from Eqs. (2) gives the following:

$$\begin{aligned} S\beta_j &= [Z_{1x}Z_{2y}(C\psi_j - 1) - Z_{1y}Z_{2y}S\psi_j - Z_{1x}Z_{2x}S\psi_j - Z_{1y}Z_{2x}(C\psi_j - 1) \\ &\quad -Z_{2y}\delta'_{jx} + Z_{2x}\delta'_{jy}]/[Z_{2x}^2 + Z_{2y}^2] \\ C\beta_j &= [-Z_{1x}Z_{2x}(C\psi_j - 1) + Z_{1y}Z_{2x}S\psi_j - Z_{1x}Z_{2y}S\psi_j - Z_{1y}Z_{2y}(C\psi_j - 1) \\ &\quad +Z_{2x}^2 + Z_{2y}^2 + Z_{2x}\delta'_{jx} + Z_{2y}\delta'_{jy}]/[Z_{2x}^2 + Z_{2y}^2] \end{aligned} \tag{8}$$

where,

$$\begin{aligned} \delta'_{jx} &= \delta_{jx} + Z_{3x}(C\gamma_j - 1) - Z_{3y}S\gamma_j \\ \delta'_{jy} &= \delta_{jy} + Z_{3y}(C\gamma_j - 1) + Z_{3x}S\gamma_j \end{aligned} \tag{9}$$

and $cos\gamma_j$, $sin\gamma_j$, $cos\psi_j$ and $sin\psi_j$ are represented by $C\gamma_j$, $S\gamma_j$, $C\psi_j$ and $S\psi_j$, respectively. The linear and angular displacements of the coupler for the synthesis of the $\mathbf{Z}_4,\mathbf{Z}_5$ dyad are obtained from Eqs. (8) and (9).

Triad 1 of Table 4 was used to obtain the precision points for the $\mathbf{Z}_4,\mathbf{Z}_5$ dyad. The precision points are listed in Table 6, and the dyads satisfying these conditions are given in Table 7. Of the four dyads, dyad 1 was the same as the links $\mathbf{Z}_1,\mathbf{Z}_2$ of the triad and therefore not useful. Each triad can be used with three useful dyads and in turn combined with one of the four dyads listed in Table 5, resulting in a

Table 4: Triads satisfying the prescribed conditions (free choice $R = 4.0$ and $\theta = 60^o$)

No.	Z_{1x}	Z_{1y}	Z_{3x}	Z_{3y}
1	-7.181273	0.623436	-7.664370	-1.761273
2	6.672161	16.31931	4.614914	14.96745
3	-0.863371	0.043467	-0.159152	-3.113884
4	-10.61403	12.97098	-14.70476	12.93994

Table 5: Dyads satisfying the prescribed conditions for motion generation problem in Table 3 (Dyad ($\mathbf{Z_6}$,$\mathbf{Z_7}$))

No.	Z_{6x}	Z_{6y}	Z_{7x}	Z_{7y}
1	-0.6922	6.0224	-1.8189	-0.9641
2	1.0795	1.0685	-0.7807	3.9692
3	-0.0309	-0.3369	-2.5263	4.4872
4	0.0362	1.0648	-2.0962	3.4017

Stephenson's six-bar mechanism satisfying the conditions prescribed in Table 3. For the arbitrarily chosen R and θ values, the dyads and triads may be combined to form 48 different six-bar mechanisms. One such six-bar mechanism formed by triad 1 of Table 4, dyad 1 of Table 5, and dyad 3 of Table 7, is depicted in Fig. 8. Also shown are the coupler in the prescribed positions and the coupler curve.

Table 6: Precision points for dyad ($\mathbf{Z}_4$,$\mathbf{Z}_5$) from triad 1 of Table 4

Precision Point	δ_{jx}	δ_{jy}	β_j deg.
2	-0.061346	-2.180604	12.98001
3	1.149173	-7.083504	-77.28882
4	2.500159	-9.201590	-94.16625
5	3.606853	-10.51808	-131.4946

Table 7: Dyads satisfying the precision points listed in Table 6

No.	Z_{4x}	Z_{4y}	Z_{5x}	Z_{5y}
1	-7.18117	0.62342	1.51694	1.07942
2	2.90116	4.81609	-3.95521	6.67584
3	1.25730	1.84760	0.32284	4.50405
4	-10.6034	1.12051	0.52894	0.23814

More triad solutions were obtainable by letting R and θ vary. First R was allowed to vary between 2 and 10 with θ fixed at 60^o. The Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y} and R values were found using continuation methods to solve 4 equations in 5 unknowns. Next R was fixed to be 2.0 and then 10.0, and θ was allowed to vary. The locations of the fixed pivot, crank pivot and coupler pivot relative to the coupler point in its initial

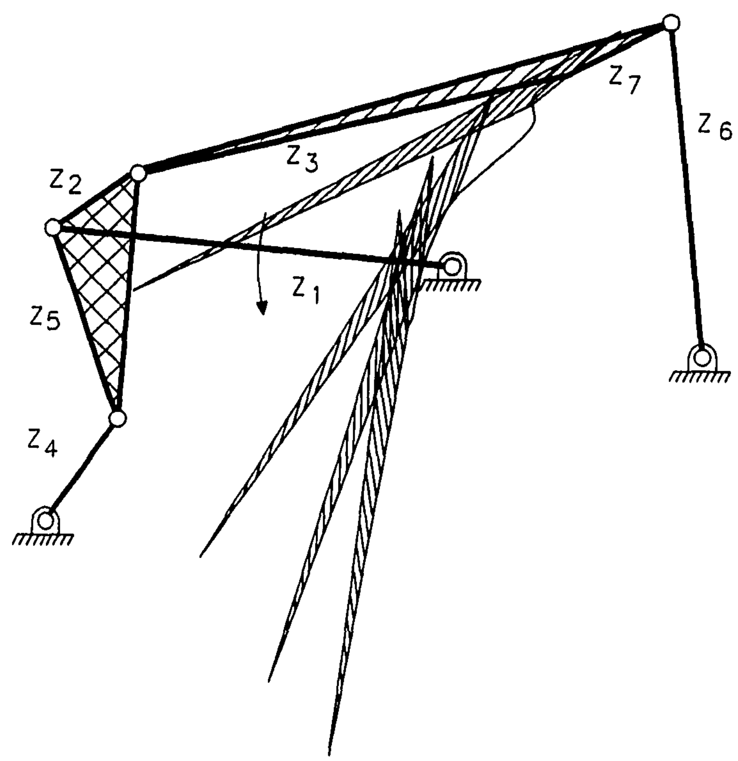

Figure 8: Six-bar mechanism formed by triad 1 of Table 4, dyad 1 of Table 5 and dyad 3 of Table 7 in its initial configuration along with the coupler curve and specified coupler orientations

position were plotted. These curves are shown in Figs. 9 and Figs. 10. The choice of R was arbitrary, and values of 2.0 and 10.0 were simply picked to demonstrate the procedure. By picking a fixed pivot location on Figs. 9 or 10, we would obtain four corresponding coupler pivot and crank pivot locations giving us the four feasible triad solutions for the chosen R and θ. Thus the continuation method solves the five position motion generation with prescribed timing problem and provides infinities of triads.

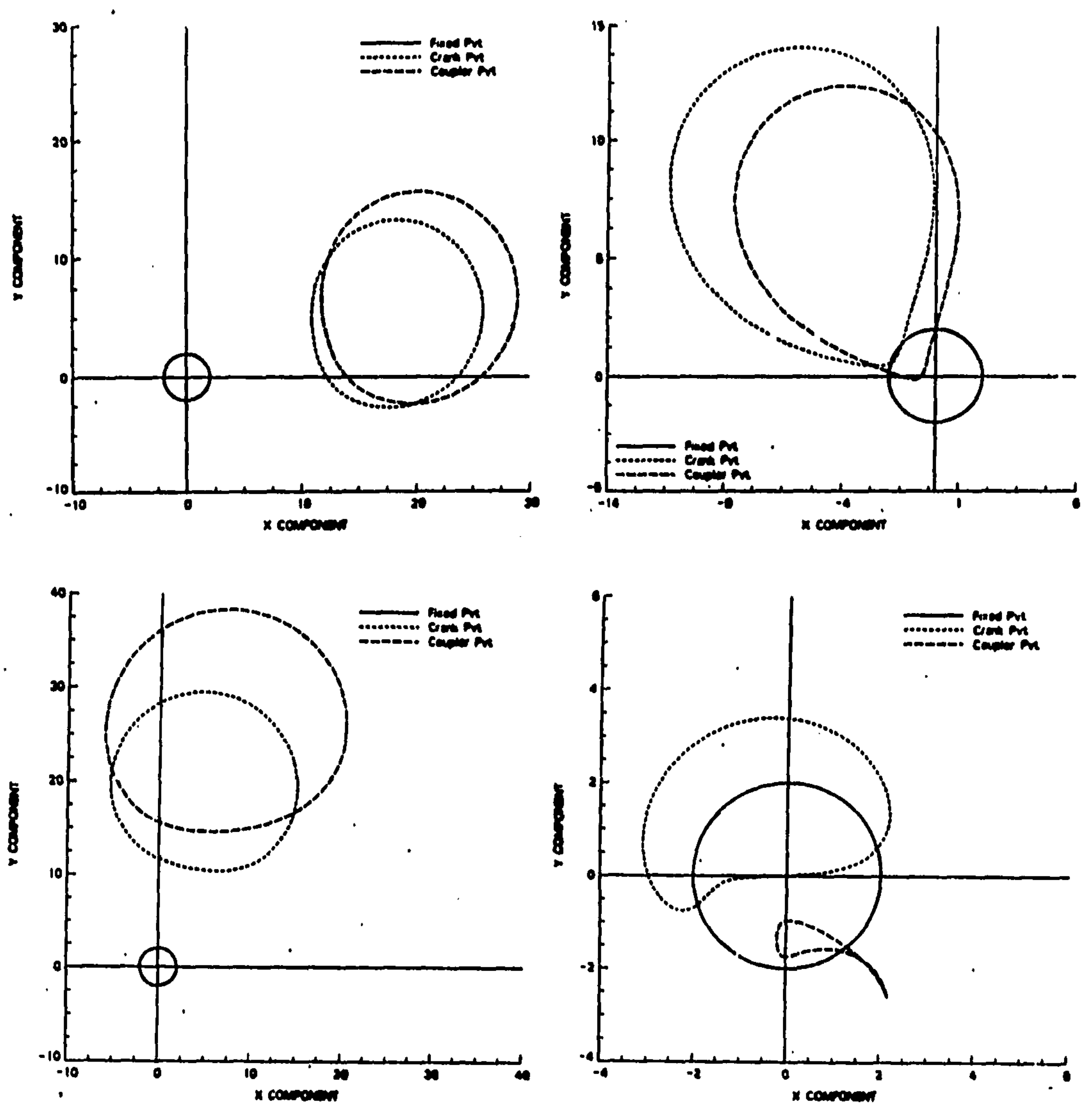

Figure 9: The triad design curves obtained for $R = 2.0$

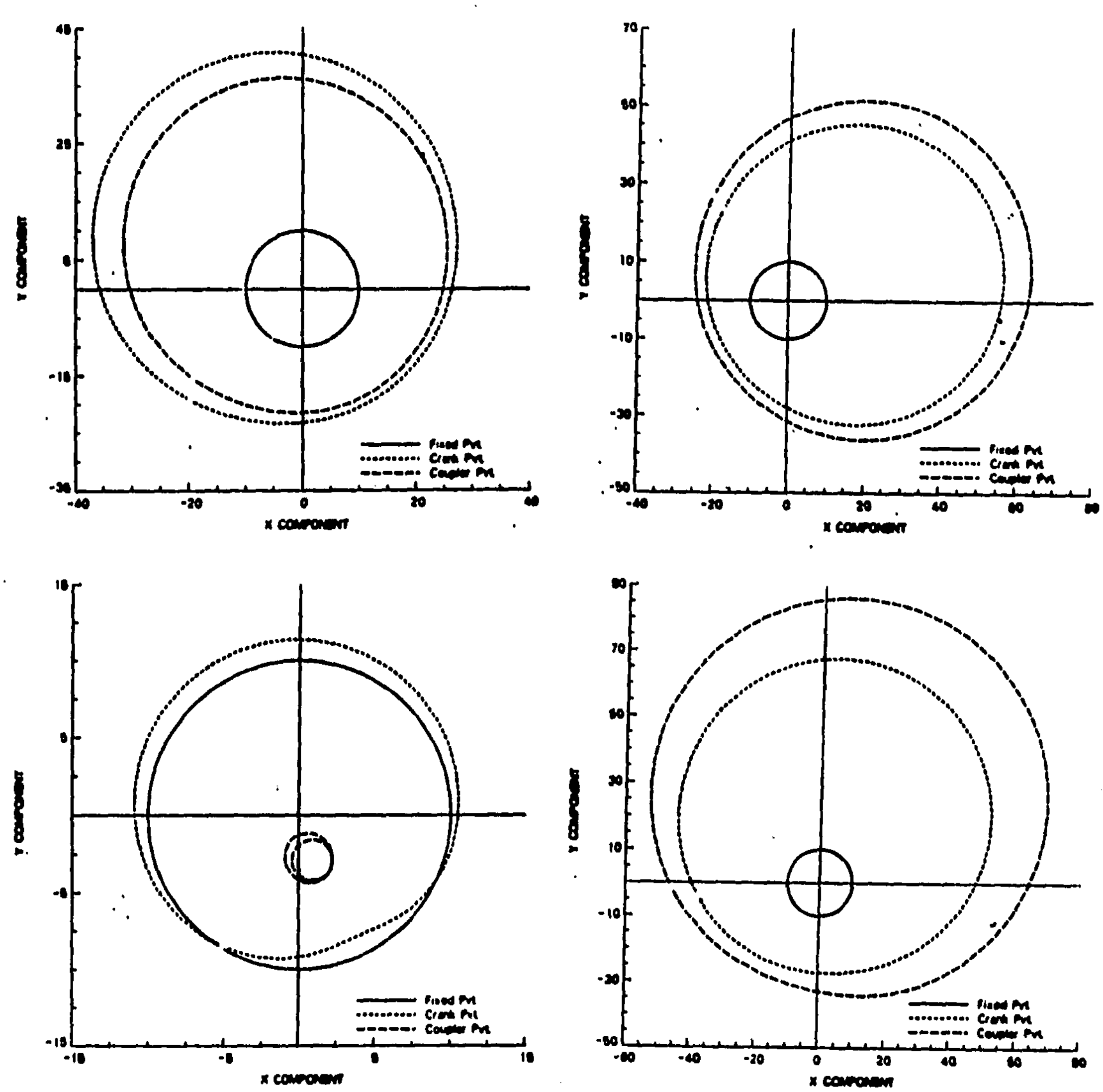

Figure 10: The triad design curves obtained for $R = 10.0$

REFERENCES

Chase, T. R., A. G. Erdman, and D. R. Riley. 1985. Improved Centerpoint Curve Generation Techniques for Four-Precision Position Synthesis using the Complex Number Approach. Journal of Mechanisms, Transmissions, and Automation in Design 107:370-376.

Chase, T. R., A. G. Erdman, and D. R. Riley. 1987. Triad Synthesis for up to Five Design Positions with Application to the Design of Arbitrary Planar Mechanisms. Journal of Mechanisms, Transmissions, and Automation in Design 109:426-434.

Erdman, A. G. 1981. Three and Four Precision Point Kinematic Synthesis of Planar Linkages. Journal of Mechanism and Machine Theory 16:227-245.

Garcia, C. B., and W. I. Zangwill. 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, New Jersey.

Lin, C. -S., and A. G. Erdman. 1987. Dimensional Synthesis of Planar Triads: Motion Generation With Prescribed Timing for Six Precision Positions. Journal of Mechanism and Machine Theory 22:411-419.

Morgan, A. 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. General Motors Research Laboratories, Warren, Michigan (GMR-3651).

Morgan, Alexander. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Morgan, A. P., and C. W. Wampler. 1989. Solving a Planar Four-Bar Design Problem using Continuation. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:409-416.

Subbian, T., and D. R. Flugrad. 1989a. Synthesis of Motion Generating Mechanisms by a Continuation Method. Proceedings of the 1st National Applied Mechanisms and Robotics Conference 89AMR-2B-2.

Subbian, T., and D. R. Flugrad. 1989b. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation-1989: Mechanical Systems

Analysis, Design and Simulation. ASME DE-19-3:425-432.

Subbian, T., and D. R. Flugrad. 1990. Six and Seven Position Triad Synthesis Using a Continuation Method. Under review for publication in the Journal of Mechanical Design.

Tsai, L. -W. and J. -J. Lu. 1989. Coupler-Point-Curve Synthesis Using Homotopy Methods. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:417-423.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990. Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics. Journal of Mechanical Design 112, no.1:59-68.

PART IV.

SIX AND SEVEN POSITION TRIAD SYNTHESIS USING CONTINUATION METHODS

INTRODUCTION

A triad consists of three links connected by two pivots and forms a part of many single and multi-loop mechanisms. Due to its usefulness in synthesis of mechanisms, the design of a triad is important. Chase et al. [1987] and Subbian and Flugrad [1990] have carried out triad synthesis for five prescribed precision points. Lin and Erdman [1987] have applied a compatibility linkage approach to obtain triad-Burmester curves for a six position synthesis problem. In their approach, Lin and Erdman varied the angular displacement of the intermediate link of the triad between 0 and 360^o and solved for the angular displacement of the third link from a sixth degree polynomial equation. Then, for a specified intermediate link angular displacement they generated the Burmester curves for the triad by analyzing a ten-bar one-degree-of-freedom compatibility linkage.

Here, the angular displacements of the intermediate link are eliminated from the equations, thereby reducing the number of equations to one per specified displacement. Therefore for six precision points, the number of equations is five, involving six unknown link dimensions. These equations are solved using continuation methods to generate the Burmester curves for the triads. When seven positions are prescribed, a finite solution set is obtained, since the number of unknowns equals the number of equations. This approach is an extension of the method used for five prescribed precision points in an earlier paper [Subbian and Flugrad, 1990].

The continuation method is a mathematical procedure to solve systems of nonlinear polynomial equations. Recently, the method has received substantial attention for kinematic synthesis problems. Subbian and Flugrad [1989a and 1989b] considered the design of four-bar mechanisms for path and motion generation. Morgan and

Wampler [1989] designed a four-bar path generation linkage for five prescribed positions, and Wampler et al. [1990a] and Tsai and Lu [1989] considered nine position synthesis. Design of geared five-bar mechanisms was the focus of the work by Roth and Freudenstein [1963] and Starns and Flugrad [1990], and six-bar mechanisms were considered by Subbian and Flugrad [1990]. Wampler et al. [1990b] have provided an overview of continuation methods and its application to kinematic synthesis.

Detailed descriptions of continuation methods have been provided by Morgan [1987] and Garcia and Zangwill [1981]. The use of projective transformations to avoid solutions at infinity was considered by Morgan [1986, 1987] and path reduction schemes have been presented by Morgan and Sommese [1987a, 1987b]. Morgan and Sommese [1989], as well as Wampler et al. [1990b] have discussed the different homotopies and their relative merits.

In the sections to follow, the displacement equations for a triad are developed using complex number notation. These equations are cast in polynomial form by eliminating the unknown angles. The equations thus obtained are solved for motion generation with prescribed timing involving six and seven precision points. The procedure is implemented in synthesizing eight-bar and geared five-bar mechanisms.

DEVELOPMENT OF EQUATIONS FOR TRIAD SYNTHESIS

The complex number approach may be used to write a loop closure equation for a triad in two finitely displaced positions as shown in Fig. 1. The loop closure equation is:

$$\mathbf{Z_1}(e^{i\psi_j} - 1) + \mathbf{Z_2}(e^{i\beta_j} - 1) - \mathbf{Z_3}(e^{i\gamma_j} - 1) = \delta_j \tag{1}$$

where the linear displacement of the coupler, δ_j, angular displacement of the coupler, γ_j, and angular displacement of the crank, ψ_j, are specified.

By breaking Eq. (1) into its real and imaginary component equations and eliminating the angular displacement of the intermediate link, β_j, we obtain [Subbian and Flugrad, 1990]

$$F_{j-1} = 2A_j sin\psi_j - 2B_j(cos\psi_j - 1) + D_j \tag{2}$$

where $j = 2$ to n, and

$$\begin{aligned}
A_j &= Z_{2x}Z_{1y} - Z_{2y}Z_{1x} + Z_{1y}\delta_{jx} - Z_{1x}\delta_{jy} + (cos\gamma_j - 1)[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}] \\
&- sin\gamma_j[Z_{1y}Z_{3y} + Z_{1x}Z_{3x}] \\
B_j &= Z_{1x}^2 + Z_{1y}^2 + Z_{2x}Z_{1x} + Z_{2y}Z_{1y} + Z_{1x}\delta_{jx} + Z_{1y}\delta_{jy} \\
&+ (cos\gamma_j - 1)[Z_{3x}Z_{1x} + Z_{3y}Z_{1y}] + sin\gamma_j[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}] \\
D_j &= 2Z_{2x}\delta_{jx} + 2Z_{2y}\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \\
&+ 2(cos\gamma_j - 1)[Z_{3x}Z_{2x} + Z_{3y}Z_{2y} + Z_{3x}\delta_{jx} + Z_{3y}\delta_{jy} - Z_{3x}^2 - Z_{3y}^2] \\
&+ 2\sin\gamma_j[Z_{3x}Z_{2y} - Z_{3y}Z_{2x} + Z_{3x}\delta_{jy} - Z_{3y}\delta_{jx}]
\end{aligned}$$

In generating the triad-Burmester curves, it is helpful to replace Z_{2x} and Z_{2y} by polar coordinates R and θ which locate the coupler point with respect to the fixed pivot in its initial configuration, as shown in Fig. 2. The relationship between R, θ and Z_{2x}, Z_{2y} is:

$$\begin{aligned} Z_{2x} &= Rcos\theta + Z_{3x} - Z_{1x} \\ Z_{2y} &= Rsin\theta + Z_{3y} - Z_{1y} \end{aligned} \tag{3}$$

Substitution of Z_{2x} and Z_{2y} into Eq. (2), followed by rearrangement yields

$$\begin{aligned} F_{j-1} &= A_{1j}RZ_{1y} + A_{2j}RZ_{1x} + A_{3j}(Z_{3x}Z_{1y} - Z_{3y}Z_{1x}) \\ &+A_{4j}(Z_{3y}Z_{1y} + Z_{3x}Z_{1x}) + A_{5j}RZ_{3x} + A_{6j}RZ_{3y} \\ &+A_{7j}Z_{1y} + A_{8j}Z_{1x} + A_{9j}R + A10jZ_{3x} + A_{11j}Z_{3y} + A_{12j} \end{aligned} \tag{4}$$

where $j = 2$ to n, and

$$\begin{aligned} A_{1j} &= 2cos\theta sin\psi_j - 2sin\theta(cos\psi_j - 1) \\ A_{2j} &= -2sin\theta sin\psi_j - 2cos\theta(cos\psi_j - 1) \\ A_{3j} &= 2cos\gamma_j sin\psi_j - 2sin\gamma_j cos\psi_j \\ A_{4j} &= -2sin\gamma_j sin\psi_j - 2cos\gamma_j cos\psi_j + 2 \\ A_{5j} &= 2(cos\gamma_j - 1)cos\theta + 2sin\gamma_j sin\theta \\ A_{6j} &= 2(cos\gamma_j - 1)sin\theta - 2sin\gamma_j cos\theta \\ A_{7j} &= 2\delta_{jx}sin\psi_j - 2\delta_{jy}cos\psi_j \\ A_{8j} &= -2\delta_{jy}sin\psi_j - 2\delta_{jx}cos\psi_j \\ A_{9j} &= 2\delta_{jx}cos\theta + 2\delta_{jy}sin\theta \\ A_{10j} &= 2\delta_{jx}cos\gamma_j + 2\delta_{jy}sin\gamma_j \\ A_{11j} &= 2\delta_{jy}cos\gamma_j - 2\delta_{jx}sin\gamma_j \\ A_{12j} &= \delta_{jx}^2 + \delta_{jy}^2 \end{aligned}$$

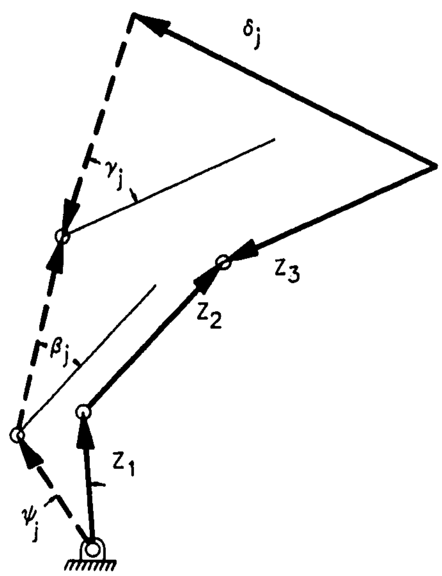

Figure 1: A triad in two finitely separated positions

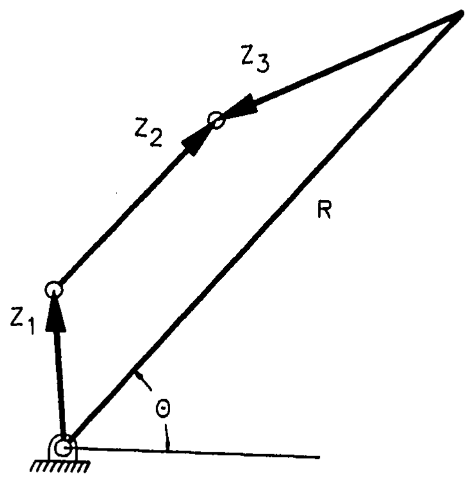

Figure 2: Triad showing the newly defined variables (R and θ)

In the above equations, the linear displacement, δ_j, of the base of vector $\mathbf{Z_3}$, angular displacement, γ_j, of the vector $\mathbf{Z_3}$, and angular displacement, ψ_j, of vector $\mathbf{Z_1}$ from the initial configuration are specified. The unknowns to be solved for are the link vector $\mathbf{Z_1}$, link vector $\mathbf{Z_3}$, radius of fixed pivot from coupler point R and its angular orientation θ. The maximum number of unknowns in Eq. (4) are 6 and so the number of specified displacements should be ≤ 6 and the limit on the number of prescribed precision points is 7.

Thus, for six position synthesis, the set of equations obtained from Eq. (4) for $j = 2, 3, 4, 5, 6$ is solved to obtain the triads. Here the number of unknowns is 6 and the number of equations is 5. An arbitrary value is assigned for θ and points on the triad-Burmester curves are determined. These points are later used to trace the Burmester curves by allowing θ to vary.

For seven position triad synthesis the set of equations obtained from Eq. (2) for $j = 2, 3, 4, 5, 6, 7$ is solved for the variables, Z_{1x}, Z_{1y}, Z_{2x}, Z_{2y}, Z_{3x}, Z_{3y}. Here, since there are 6 equations involving 6 unknowns, a finite solution set is obtained.

SIX POSITION SYNTHESIS

For six position synthesis, the set of five equations determined from Eq. (4) (as described above) was solved by continuation. The unknowns Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y}, and R were determined for a specified value of θ to obtain points on the Burmester curves. Then θ was varied to generate the Burmester curves.

To solve the system (five equations in five unknowns), CONSOL 8 [Morgan, 1987] was used. A predictor-corrector type path tracker is used in this program in which the prediction step is taken along the direction of the tangent to the continuation curve, and Newton's method is used to correct the values back to the curve. Gaussian elimination is used to determine the direction of the tangent and to implement Newton's method. To trace the Burmester curves (to solve five equations in six unknowns), first order differential equations of the unknowns with respect to θ were determined from the extended Jacobian of the system using Cramer's rule [Subbian and Flugrad, 1989b, and Morgan, 1981]. These differential equations were solved using a variable step Adam's method and employing as initial conditions the real solutions of the CONSOL 8 run, where the θ value was fixed.

On fixing θ, the degree of each of the equations in the system is two, and thus the total degree of the system is $2^5 = 32$. If traditional 1-homogeneous homotopy is used, 32 paths have to be tracked. If 3-homogeneous homotopy is used, the Bezout number of the system is 30 [Morgan and Sommese, 1987a, 1987b] indicating a reduction of only two in the number of paths (refer to Appendix A). Therefore, a traditional 1-homogeneous homotopy was used to obtain the solutions after implementing a projective transformation to avoid solutions at infinity [Morgan, 1986, 1987]. The following homotopy was used:

$$\mathbf{H}(y,t) = (1-t)\mathbf{G}(y) + t\mathbf{F}(y) \tag{5}$$

where $G_j(y) = p_j^{d_j} y^{d_j} - q_j^{d_j} = 0$, d_j is the degree of equation F_j, and p_j and q_j are randomly chosen constants.

Random complex displacements δ_{jx}, δ_{jy}, γ_j and ψ_j were chosen and Eqs. (4) solved for Z_{1x}, Z_{1y}, Z_{3x}, Z_{3y} and R on fixing θ to be a random complex value. Of the 32 paths tracked, 17 resulted in solutions at infinity, and 15 paths produced complex solutions. Sixteen of the solutions at infinity were singular and of the three forms: $(0,0,Z_{3x},Z_{3y},0)$, $(Z_{1x},Z_{1y},0,0,0)$, or $(Z_{1x},Z_{1y},Z_{3x},Z_{3y},0)$. The solutions found in the first and second forms were of multiplicity four, and those in the third form were of multiplicity eight. The unique solution was of a fourth form $(0,0,0,0,R)$.

All solutions at infinity should satisfy the homogeneous part of Eqs. (4), which is:

$$\begin{aligned} F^o_{j-1} &= A_{1j}RZ_{1y} + A_{2j}RZ_{1x} + A_{3j}(Z_{3x}Z_{1y} - Z_{3y}Z_{1x}) \\ &\quad + A_{4j}(Z_{3y}Z_{1y} - Z_{3x}Z_{1x}) + A_{5j}RZ_{3x} + A_{6j}RZ_{3y} = 0 \end{aligned} \tag{6}$$

It can be easily shown that the solutions of the first, second and fourth forms noted above satisfy Eq. (6) and are coefficient independent (A_{1j} through A_{6j}). For those solutions of the third form to satisfy Eq. (6), independent of the coefficients, $(Z_{3x}Z_{1y} - Z_{3y}Z_{1x})$ and $(Z_{3y}Z_{1y} - Z_{3x}Z_{1x})$ must be zero. This was observed in all eight solutions involved.

Therefore, all 17 solutions found at infinity will always be at infinity (irrespective of the specified displacements and θ); these solutions can be eliminated by using a secant homotopy (refer to Appendix B) of the following form:

$$\mathbf{H}(y,t) = \mathbf{F}(A^s_{mj}, y) = 0 \tag{7}$$

where $m = 1, 2, ..., 12$, $j = 2, 3, ..., 6$ and

$$A^s_{mj} = A^o_{mj} e^{i\sigma}(1-t) + A^1_{mj} t \tag{8}$$

in which A^o_{mj} are the coefficients of Eqs. (4) for randomly chosen complex displacements (solved before), and A^1_{mj} are the coefficients for the specified displacements. Also, in Eq. (8), σ is a random angle. Therefore, the random complex displacements and selected θ used for the 1-homogeneous homotopy gave the start system for the secant homotopy; the 15 complex solutions obtained provided the starting points for the solution process.

By employing secant homotopy, the CPU time declined from 7:01 minutes (for 1-homogeneous homotopy) to 3:48 minutes in one specific example. Parameter homotopy [Morgan and Sommese, 1989] was also implemented, resulting in more CPU time than that for traditional homotopy (refer to Appendix B for parameter homotopy). Therefore, secant homotopy was particularly advantageous. Of the 15 solutions possible, at least one real solution was expected (since complex solutions appear with their conjugates). However, real solutions produced may be at infinity, depending on the choice of displacements and θ.

The real solutions obtained are discrete points on the triad-Burmester curves. The solution curves are then traced by allowing θ to vary, as mentioned previously. These curves represent the fixed pivot, crank pivot and coupler pivot locations relative to the coupler point with the triad in its initial position.

Table 1: Precision points for the design of eight-bar mechanism

Precision Point	δ_{jx}	δ_{jy}	γ_j deg.	ψ_j deg.
2	-2.0	0.0	18.0	30.0
3	-4.0	0.0	36.0	60.0
4	-6.0	0.0	54.0	90.0
5	-8.0	0.0	72.0	120.0
6	-10.0	0.0	90.0	150.0

Table 2: Triads satisfying the prescribed conditions listed in Table 1 for $\theta = 60^o$

No.	Z_{1x}	Z_{1y}	Z_{3x}	Z_{3y}	R
1	-13.87760	-6.783088	-28.40739	-1.788351	16.42879
2	-4.958358	-1.382021	-10.14451	-10.06671	10.30736
3	-6.637034	-1.778388	-14.18740	-14.18740	10.0
4	-4.651096	-1.246257	-9.677005	-9.677005	10.0
5	-27.11246	-3.786862	-120.9407	174.1207	25.47498
6	-21.13258	-10.99244	-86.76292	69.99115	22.18998

Example: Synthesis of an eight-bar mechanism

An eight-bar mechanism, shown in Fig. 3, was synthesized by considering it as three triads pieced together. The individual triads were designed to satisfy the conditions prescribed in Table 1 for the mechanism. The angular position, θ, of the coupler point from the fixed pivot was chosen to be 60^o. Of the fifteen solutions obtained, six were real, eight were complex, and one was at infinity. Table 2 lists the six real solutions, and Fig. 4 confirms that triad 1 of Table 2 satisfies the prescribed conditions.

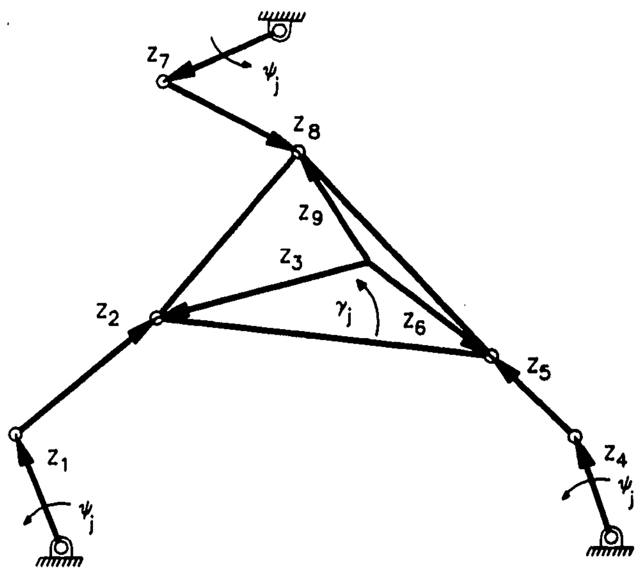

Figure 3: Eight-bar mechanism

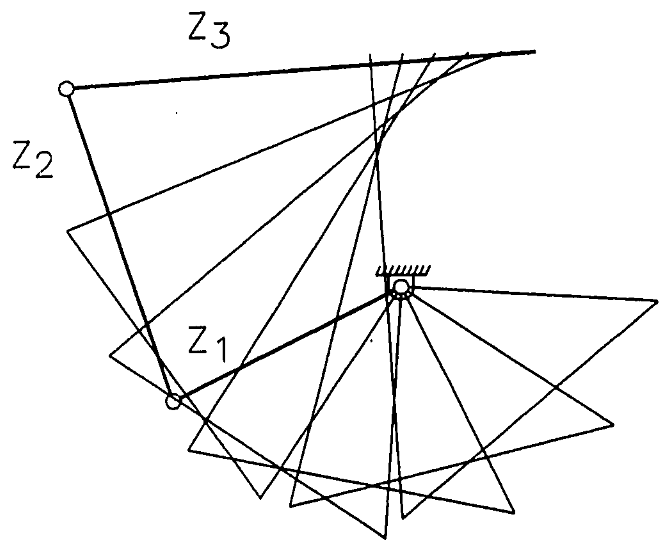

Figure 4: Triad 1 of Table 2 in the six prescribed positions

Burmester curves were traced from the solutions listed in Table 2. Two sets of curves were obtained as shown in Figs. 5 and 6. These curves locate the fixed pivot, crank pivot and coupler pivot (from the coupler point) for the triad in its first position. Each point on the fixed pivot curve correspond to a distinct point on the crank pivot and coupler pivot curves, thereby providing the required triad. The triad shown in Fig. 4 was combined with two other triads (obtained from the Burmester curves) to obtain the eight-bar mechanism shown in Fig. 7. Here we see the designed mechanism in its initial configuration along with the coupler in its six prescribed positions.

A couple of important points should be emphasized at this juncture. First, the generated solution curves (Burmester curves) might depend on the values chosen for θ. This is not true in all cases, but is a distinct possibility, since θ is varied indirectly through a path variable, s. Variation of s does not ensure that we will obtain the solutions for all θ, ranging from 0 to 2π. For a value of θ, say $\pi/2$, the solutions might head toward infinity or become complex, so we are unable to trace a portion of the curve. Normally, the designer will restrict his search to a certain range of θ and so this might not be an important factor.

The second point is that θ need be varied only between 0 and π since, due to the method used to define R and θ, solutions for $\theta + \pi$ are the same as those at θ with a negative R.

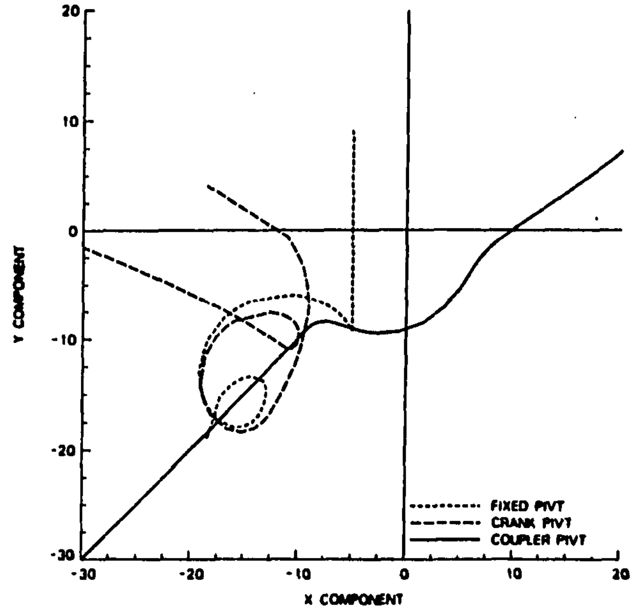

Figure 5: Triad Burmester curves for the precision points given in Table 1 (curve 1)

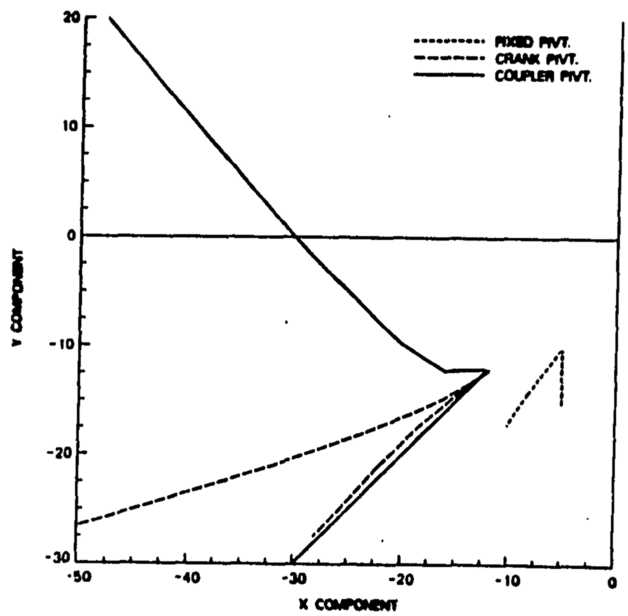

Figure 6: Triad Burmester curves for the precision points given in Table 1 (curve 2)

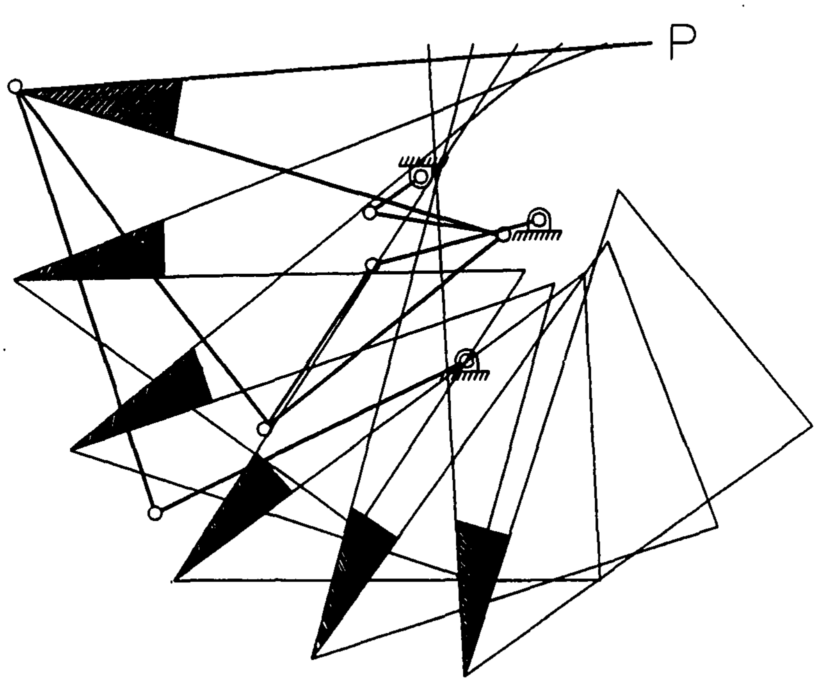

Figure 7: Eight-bar mechanism in its initial configuration along with the coupler in the subsequent positions

SEVEN POSITION SYNTHESIS

For a seven position synthesis problem, six displacements are specified. In this case, it is easier to apply secant homotopy and also to identify the solutions at infinity for Eq. (2) rewritten as follows:

$$
\begin{aligned}
F_{j-1} \;=\; & b_{1j}(Z_{3x}^2 + Z_{3y}^2) + b_{2j}(Z_{1x}^2 + Z_{1y}^2) + b_{3j}(Z_{2x}Z_{3x} + Z_{2y}Z_{3y}) + \\
& b_{4j}(Z_{2y}Z_{3x} - Z_{2x}Z_{3y}) + b_{5j}(Z_{1x}Z_{3x} + Z_{1y}Z_{3y}) + \\
& b_{6j}(Z_{1y}Z_{3x} - Z_{1x}Z_{3y}) + b_{7j}(Z_{1x}Z_{2x} + Z_{1y}Z_{2y}) + \\
& b_{8j}(Z_{1y}Z_{2x} - Z_{1x}Z_{2y}) + b_{9j}Z_{3x} + b_{10j}Z_{3y} \\
& + b_{11j}Z_{2x} + b_{12j}Z_{2y} + b_{13j}Z_{1x} + b_{14j}Z_{1y} + b_{15j} \qquad (9)
\end{aligned}
$$

where $j = 2, 3, 4, 5, 6, 7$ and,

$$
\begin{aligned}
b_{1j} &= -2(cos\gamma_j - 1) \\
b_{2j} &= -2(cos\psi_j - 1) \\
b_{3j} &= -b_{1j} \\
b_{4j} &= 2sin\gamma_j \\
b_{5j} &= -2[(cos\gamma_j - 1)(cos\psi_j - 1) + sin\gamma_j sin\psi_j] \\
b_{6j} &= 2[(cos\gamma_j - 1)sin\psi_j - (cos\psi_j - 1)sin\gamma_j] \\
b_{7j} &= b_{2j} \\
b_{8j} &= 2sin\psi_j \\
b_{9j} &= 2[(cos\gamma_j - 1)\delta_{jx} + sin\gamma_j\delta_{jy}] \\
b_{10j} &= 2[(cos\gamma_j - 1)\delta_{jy} - sin\gamma_j\delta_{jx}] \\
b_{11j} &= 2\delta_{jx} \\
b_{12j} &= 2\delta_{jy} \\
b_{13j} &= -2[(cos\psi_j - 1)\delta_{jx} + sin\psi_j\delta_{jy}] \\
b_{14j} &= 2[sin\psi_j\delta_{jx} - (cos\psi_j - 1)\delta_{jy}] \\
b_{15j} &= \delta_{jx}^2 + \delta_{jy}^2
\end{aligned}
$$

The degree of each of Eqs. (9) is two. Therefore, the total degree of the system is $2^6 = 64$. On expressing the system in 2- or 3- homogeneous forms, the Bezout number of the system is greater than 64. Hence, a 1-homogeneous form was used to solve the equations for which 64 paths had to be tracked. CONSOL 8 [Morgan, 1987] was used. The start system and homotopy given by Eq. (5) was applied after carrying out a projective transformation.

Random complex displacements δ_j, γ_j and ψ_j were chosen, and Eqs. (9) were solved to identify the number of solutions at infinity and their form. The number of coefficient independent solutions at infinity was 47. These solutions were in the following five forms: $(0, 0, Z_{2x}, Z_{2y}, 0, 0)$, $(Z_{1x}, -iZ_{1x}, Z_{2x}, -iZ_{2x}, Z_{3x}, -iZ_{3x})$, $(Z_{1x}, iZ_{1x}, Z_{2x}, iZ_{2x}, Z_{3x}, iZ_{3x})$, $(Z_{1x}, Z_{1y}, -Z_{1x}, -Z_{1y}, 0, 0)$, or $(0, 0, Z_{2x}, Z_{2y}, Z_{2x}, Z_{2y})$. These solutions should satisfy the homogeneous part of Eqs. (9), which is:

$$\begin{aligned} F_{j-1}^{o} &= b_{1j}(Z_{3x}^2 + Z_{3y}^2) + b_{2j}(Z_{1x}^2 + Z_{1y}^2) + b_{3j}(Z_{2x}Z_{3x} + Z_{2y}Z_{3y}) + \\ &\quad b_{4j}(Z_{2y}Z_{3x} - Z_{2x}Z_{3y}) + b_{5j}(Z_{1x}Z_{3x} + Z_{1y}Z_{3y}) + \\ &\quad b_{6j}(Z_{1y}Z_{3x} - Z_{1x}Z_{3y}) + b_{7j}(Z_{1x}Z_{2x} + Z_{1y}Z_{2y}) + \\ &\quad b_{8j}(Z_{1y}Z_{2x} - Z_{1x}Z_{2y}) \end{aligned} \tag{10}$$

It is easily shown that forms 1, 2 and 3 above satisfy Eqs. (10) independent of the coefficients (b_{ij}'s). Forms 4 and 5 are verified for $b_{7j} = b_{2j}$ and $b_{3j} = -b_{1j}$.

Of the 64 paths tracked for random complex displacements, 47 solutions at infinity and 17 complex solutions were obtained. Among the solutions at infinity, forms 1, 4 and 5 had a multiplicity of 3, and solution forms 2 and 3 had a multiplicity of 19. These solutions at infinity have no physical meaning, as they are coefficient independent and singular. Also, the 47 solutions at infinity are always present since they are not dependent on the choice of displacements. A secant homotopy (refer to Appendix B) was used to eliminate these 47 solutions and to obtain the useful solutions by tracking the remaining 17 complex solutions.

A secant homotopy was implemented with the system of equations solved using random displacements as the start system. The homotopy function had the following form:

$$\mathbf{H}(Zmn, t) = \mathbf{F}(b_{ij}^{s}, Zmn) \tag{11}$$

where, $m = 1, 2, 3$; $n = x, y$; $i = 1, 2, ..., 15$; $j = 2, 3, ..., 7$ and

$$b_{ij}^{s} = (1 - t)e^{i\sigma} b_{ij}^{o} + t b_{ij}^{1} \tag{12}$$

in which, b_{ij}^{o} are coefficients of Eqs. (9) for displacements chosen randomly and b_{ij}^{1} are coefficients of Eqs. (9) for the prescribed displacements. The parameter σ is a random angle. The CPU for one particular run was 24.3 minutes for a 1-homogeneous homotopy, and 9.7 minutes for a secant homotopy.

In the following section, the secant homotopy formulation is used to design a geared five-bar mechanism for seven position path generation.

Example: Synthesis of a geared five-bar mechanism

A geared five-bar mechanism shown in Fig. 8 was designed for path generation. The mechanism can be designed by considering the dyad and triad independently. The $\mathbf{Z_4}$,$\mathbf{Z_5}$ dyad can be assigned arbitrary values and the inverse kinematics of the dyad can be carried out (as if it were a planar two link robot) to determine the angular positions γ_j and ϕ_j of the links so that the end of link $\mathbf{Z_5}$ will pass through the specified precision points.

Table 3: Precision points for the design of geared five-bar mechanism

Precision Point	δ_{jx}	δ_{jy}
1	5.0	5.0
2	4.5980762	6.5
3	3.5	7.5980762
4	2.0	8.0
5	1.0	8.0
6	0.0	8.0
7	-1.0	8.0

The linear displacements, δ_j's, of the coupler are specified and its angular displacements, γ_j's, are determined by analyzing the dyad. The angular displacements of the links $\mathbf{Z}_1$ and $\mathbf{Z}_4$ are related through the gear ratio. With the ϕ_j values from the inverse kinematic analysis of the dyad, the ψ_j values of the triad are determined. By synthesizing a triad which satisfies these conditions and pairing it with the dyad, the geared five-bar mechanism that satisfies the prescribed conditions is obtained.

A geared five-bar mechanism was designed to pass through the precision points listed in Table 3. The length of the dyad members were chosen to be 5.0 and 6.0 for $\mathbf{Z}_4$ and $\mathbf{Z}_5$ respectively. The inverse kinematics of the dyad was carried out for this dyad to obtain the angular positions of the links as the mechanism passes through the prescribed points. One set of angular displacements obtained (two sets are possible) is listed in Table 4. The displacements to be satisfied by the triad are obtained by subtracting the initial linear and angular positions from the subsequent positions. Table 5 lists the precision points obtained for a gear ratio of -1.

Table 4: Angular disp. obtained from inverse kinematics of dyad ($\mathbf{Z_4}$,$\mathbf{Z_5}$)

No.	ϕ_j deg.	γ_j deg.
1	101.5270	0.96307
2	103.5743	15.8594
3	110.4334	29.0418
4	122.2365	38.9362
5	130.8243	44.6469
6	138.5092	51.3752
7	145.0743	58.8969

Table 5: Precision points to be satisfied by the triad

Precision Point	δ_{jx}	δ_{jy}	$\Delta\gamma_j$ deg.	$\Delta\psi_j$ deg.
2	-0.4019	1.5	14.8964	-2.04730
3	-1.50	2.5981	28.0787	-8.90640
4	-3.0	3.0	37.9732	-20.7095
5	-4.0	3.0	43.6838	-29.2973
6	-5.0	3.0	50.4121	-36.9822
7	-6.0	3.0	57.9338	-43.5473

Table 6: Triads satisfying the prescribed conditions listed in Table 5

No.	Z_{1x}	Z_{1y}	Z_{2x}	Z_{2y}	Z_{3x}	Z_{3y}
1	3.3472	2.0614	5.9650	14.179	-6.3007	-0.7672
2	10.214	-7.5964	-1.3062	-1.5409	-4.2831	2.1192
3	7.7752	-0.5528	0.6729	-2.4586	-6.8368	2.9584
4	1.7989	-5.0683	3.4770	-0.3569	-3.5568	-0.4093
5	1.9698	-1.3611	0.1294	-0.4160	-5.1192	-1.8455
6	2.3679	-1.0150	0.1498	-0.5908	-5.0484	-2.3776
7	-1.5815	-0.2727	-6.0948	9.6426	-6.2174	0.2629
8	-0.0256	0.0054	-1.0450	4.9773	-6.0012	-0.0877
9	-4.5546	-2.2067	942.10	-459.17	-5.9882	-0.1059

Secant homotopy was used to synthesize the triad satisfying the conditions given in Table 5. Of the 17 starting points, 9 solutions were real and 8 were complex. Only the real solutions listed in Table 6 are useful. Even among those, triads 8 and 9 are not useful because they degenerate to the $\mathbf{Z}_4$, $\mathbf{Z}_5$ dyad. Triad 1 of Table 6 coupled with the dyad is shown in Fig. 9 in its initial position along with the coupler orientations in the subsequent positions. Similarly, each of triads 2 through 7 can be combined with the dyad to obtain a geared five-bar mechanism which passes through the seven points specified in Table 3.

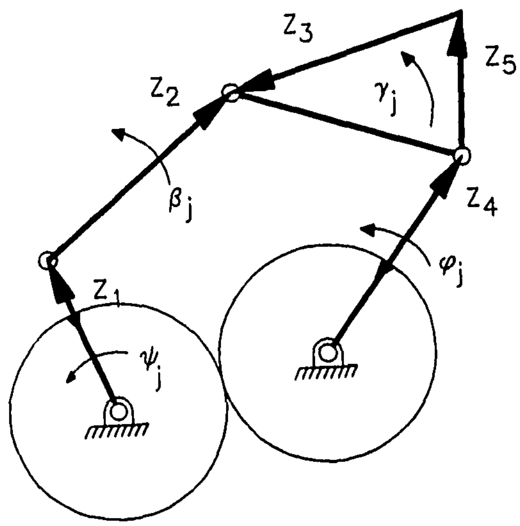

Figure 8: Geared five-bar mechanism

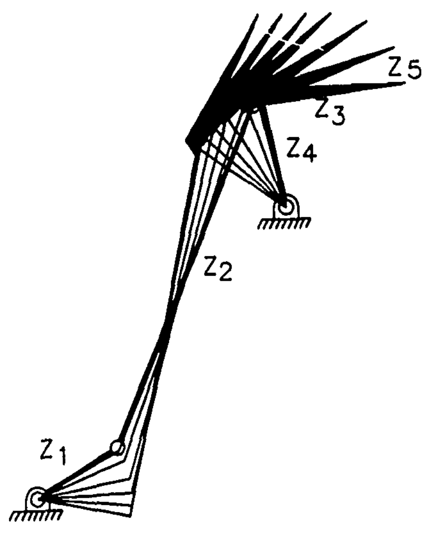

Figure 9: Five-bar mechanism obtained by combining triad 1 of Table 6 with the arbitrarily chosen dyad

REFERENCES

Chase, T. R., A. G. Erdman, and D. R. Riley. 1987. Triad Synthesis for up to Five Design Positions With Application to the Design of Arbitrary Planar Mechanisms. Journal of Mechanisms, Transmissions, and Automation in Design 109:426-434.

Garcia, C. B., and W. I. Zangwill. 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, New Jersey.

Lin, C. -S., and A. G. Erdman. 1987. Dimensional Synthesis of Planar Triads: Motion Generation With Prescribed Timing for Six Precision Positions. Journal of Mechanism and Machine Theory 22:411-419.

Morgan, A. 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. General Motors Research Laboratories, Warren, Michigan (GMR-3651).

Morgan, A. P. 1986. A Transformation to Avoid Solutions at Infinity for Polynomial Systems. Applied Mathematics and Computation 18:77-86.

Morgan, A. P. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Morgan, A. P., and A. J. Sommese. 1987a. Computing All Solutions to Polynomial Systems Using Homotopy Continuation. Applied Mathematics and Computation 24:115-138.

Morgan, A. P., and A. J. Sommese. 1987b. A Homotopy for Solving General Polynomial Systems That Respects m-Homogeneous Structures. Applied Mathematics and Computation 24:101-113.

Morgan, A. P., and A. J. Sommese. 1989. Coefficient-Parameter Polynomial Continuation. Applied Mathematics and Computation 29:123-160.

Morgan, A. P., and C. W. Wampler. 1989. Solving a Planar Four-Bar design Problem Using Continuation. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:409-416.

Roth, B., and F. Freudenstein. 1963. Synthesis of Path-Generating Mechanisms by Numerical Methods. ASME Journal of Engineering for Industry 85B, no.3:298-306.

Starns, K. G., and D. R. Flugrad. 1990. Five-Bar Path Generation Synthesis by a Continuation Method. Mechanism Synthesis and Analysis. ASME DE-25:353-359.

Subbian, T., and D. R. Flugrad. 1989a. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:425-432.

Subbian, T., and D. R. Flugrad. 1989b. Synthesis of Motion Generating Mechanisms by a Continuation Method. Proceedings of the 1st National Applied Mechanisms and Robotics Conference 89AMR-2B-2.

Subbian, T., and D. R. Flugrad. 1990. Five Position Triad Synthesis With Applications to Four- and Six-Bar Mechanisms. To appear in the Journal of Mechanical Design.

Tsai, L. W., and J. J. Lu. 1989. Coupler-Point-Curve Synthesis Using Homotopy methods. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:417-424.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990a. Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages. Mechanism Synthesis and Analysis. ASME DE-25:361-368.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990b. Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics. Journal of Mechanical Design 112:59-68.

PART V.

USE OF CONTINUATION METHODS FOR KINEMATIC SYNTHESIS

INTRODUCTION

Kinematic synthesis of linkages is restricted to a few precision points and mechanism configurations due partly to the lack of adequate numerical procedures. The kinematic equations can be developed for these problems, and the maximum number of precision points can be determined on the basis of the number of unknowns. Closed-form solutions for these equations are impossible for many synthesis problems which necessitates the use of numerical techniques. The numerical procedures currently popular, Newton-Raphson, conjugate gradient, generalized reduced gradient, and steepest descent, need good solution estimates to start the process and will not assure a complete solution set. This calls for a continuation method which does not require any initial guesses and provides all the solutions for systems of polynomial equations. As many kinematic synthesis problems can be expressed in polynomial form, continuation can be effectively used to solve them.

An example which illustrates the above concept is the synthesis of a four-bar path generation mechanism for five precision positions. Continuation methods can effectively solve this problem providing a solution set in its entirety as demonstrated by Subbian and Flugrad [1989a] as well as Morgan and Wampler [1989].

Application of a type of continuation method for kinematic synthesis was first considered by Roth and Freudenstein [1963]. They termed their technique the *bootstrap* method. Since then until recently, no further effort has been focused on its use for mechanism design due to its complexity and the lack of adequate computing facilities. Researchers such as Morgan and Sommese [1989] , and Garcia and Zangwill [1981] have made the procedure more efficient and easier to implement. These advancements and the developments in computer technology have made the application of continuation methods for kinematic synthesis feasible.

Subbian and Flugrad [1989a, 1989b, 1990] have applied the procedure to synthesize planar four-bar path, motion, and function generation mechanisms and planar six-bar motion generation mechanisms with prescribed timing. Wampler et al. [1990a] have presented an overview of various homotopies and their application for kinematic synthesis. Morgan and Wampler [1989] solved the planar four-bar path generation problem for five precision positions and Tsai and Lu [1989] extended it to nine positions using a cheater's homotopy. Wampler et al. [1990b] obtained a complete solution set for the nine position problem using a traditional coefficient parameter homotopy. Starns and Flugrad [1990] considered geared five-bar mechanisms in their work. Therefore, in the past two years a considerable amount of work has been carried out to establish the fact that continuation methods can be effectively used for kinematic synthesis of mechanisms.

The objective of this paper is to provide a detailed illustration of the steps involved in solving a kinematic synthesis problem using continuation. This is achieved through a seven position triad synthesis example and a five position revolute-spherical (R-S) dyad synthesis.

THE CONTINUATION METHOD

Continuation is a mathematical procedure that can be used to solve systems of polynomial equations. A detailed description of the procedure is provided in Morgan [1987], Garcia and Zangwill [1981], and Morgan [1986a]. To solve a system of polynomial equations, $\mathbf{F}(x) = 0$, we choose a simple system, $\mathbf{G}(x) = 0$, which is easy to solve. We then define a homotopy function by combining these two sets of equations using the homotopy parameter, t, as follows:

$$\mathbf{H}(x,t) = (1-t)\mathbf{G}(x) + t\mathbf{F}(x) = 0 \qquad (1)$$

Depending on the choice of $\mathbf{G}(x)$ and the manner in which it is combined with $\mathbf{F}(x)$, we can term the homotopy as a coefficient homotopy, parameter homotopy or secant homotopy.

In Eq.(1), as t is incremented from 0 to 1, we move away from the solutions of the start system, $\mathbf{G}(x) = 0$, to that of the target system, $\mathbf{F}(x) = 0$. The number of paths tracked will be equal to the total degree of the system [Morgan, 1987]. The solutions of the target system may take on either finite or infinite values. Projective transformation can be used to transform the solutions at infinity to be finite [Morgan, 1986b, 1987]. Some of the extraneous paths, which result in solutions at infinity, can be eliminated by taking advantage of the m-homogeneous structure of the target system. Morgan and Sommese [1987a, 1987b] discuss the procedure to identify the structure of the equations and to utilize projective transformation to avoid solutions at infinity for such systems.

Moreover, using a secant homotopy or parameter homotopy, we can eliminate the coefficient independent solutions at infinity for the m-homogeneous systems. Morgan and Sommese [1989] and Wampler et al. [1990a] consider the various homotopies available and their advantages and disadvantages.

In sections to follow, continuation methods will be utilized to solve two kinematic synthesis problems, one planar and one spatial. A detailed description of the procedure along with the path reduction technique used to minimize the number of paths is also provided.

SEVEN POSITION TRIAD SYNTHESIS

Synthesis of single and multi-loop mechanisms can be carried out by considering them as combinations of dyads and triads. For example, a triad might be utilized for motion generation with prescribed timing. Fig. 1 shows a triad in two finitely separated positions. The input angles, ψ_j's, the coupler angles, γ_j's, and the displacements, δ_j's, are to be specified, and the link vectors, $\mathbf{Z_1}$, $\mathbf{Z_2}$, and $\mathbf{Z_3}$, are to be determined. Synthesis of such triads for five prescribed positions has been considered by Chase et al. [1987] as well as by Subbian and Flugrad [1990]. Lin and Erdman [1987] have addressed the six precision position case, and the seven position synthesis problem is considered here.

Development of equations

Equations should be developed and expressed in polynomial form because continuation assures a complete solution set only for polynomials. The loop closure equation for the triad can be written using complex numbers involving the unknowns shown in Fig. 1. One complex equation or two scalar equations are obtained for each displacement from the initial configuration. The intermediate angle, β_j, can be eliminated between the two scalar equations to obtain a single scalar equation for each displacement. Eq. (2) gives the displacement equation, the development of which is discussed in Subbian and Flugrad [1990].

$$F_{j-1} \quad = \quad 2A_j sin\psi_j - 2B_j(cos\psi_j - 1) + D_j \tag{2}$$

where $j = 2, 3, 4, 5, 6, 7$ and

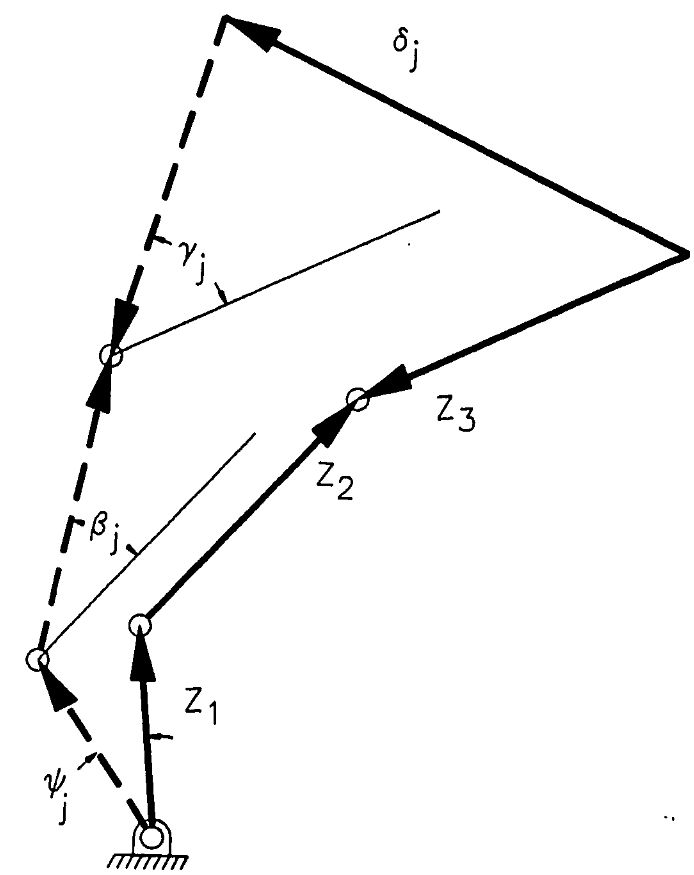

Figure 1: A triad in two finitely displaced positions

$$\begin{aligned}
A_j &= Z_{2x}Z_{1y} - Z_{2y}Z_{1x} + Z_{1y}\delta_{jx} - Z_{1x}\delta_{jy} + (cos\gamma_j - 1)[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}] \\
&\quad -sin\gamma_j[Z_{1y}Z_{3y} + Z_{1x}Z_{3x}] \\
B_j &= Z_{1x}^2 + Z_{1y}^2 + Z_{2x}Z_{1x} + Z_{2y}Z_{1y} + Z_{1x}\delta_{jx} + Z_{1y}\delta_{jy} \\
&\quad +(cos\gamma_j - 1)[Z_{3x}Z_{1x} + Z_{3y}Z_{1y}] + sin\gamma_j[Z_{3x}Z_{1y} - Z_{3y}Z_{1x}] \\
D_j &= 2Z_{2x}\delta_{jx} + 2Z_{2y}\delta_{jy} + \delta_{jx}^2 + \delta_{jy}^2 \\
&\quad +2(cos\gamma_j - 1)[Z_{3x}Z_{2x} + Z_{3y}Z_{2y} + Z_{3x}\delta_{jx} + Z_{3y}\delta_{jy} - Z_{3x}^2 - Z_{3y}^2] \\
&\quad +2\sin\gamma_j[Z_{3x}Z_{2y} - Z_{3y}Z_{2x} + Z_{3x}\delta_{jy} - Z_{3y}\delta_{jx}]
\end{aligned}$$

For a seven position synthesis problem, six sets of linear and angular displacements are specified. Six equations are thus obtained for the specified displacements which are used to solve for the six unknowns, Z_{1x}, Z_{1y}, Z_{2x}, Z_{2y}, Z_{3x} and Z_{3y}. This is the maximum number of displacements that can be specified for precision position synthesis.

Eq. (2) is rewritten as follows to make it easier to apply secant and parameter homotopies:

$$\begin{aligned}
F_{j-1} &= b_{1j}(Z_{3x}^2 + Z_{3y}^2) + b_{2j}(Z_{1x}^2 + Z_{1y}^2) + b_{3j}(Z_{2x}Z_{3x} + Z_{2y}Z_{3y}) \\
&\quad +b_{4j}(Z_{2y}Z_{3x} - Z_{2x}Z_{3y}) + b_{5j}(Z_{1x}Z_{3x} + Z_{1y}Z_{3y}) \\
&\quad +b_{6j}(Z_{1y}Z_{3x} - Z_{1x}Z_{3y}) + b_{7j}(Z_{1x}Z_{2x} + Z_{1y}Z_{2y}) \\
&\quad +b_{8j}(Z_{1y}Z_{2x} - Z_{1x}Z_{2y}) + b_{9j}Z_{3x} + b_{10j}Z_{3y} \\
&\quad +b_{11j}Z_{2x} + b_{12j}Z_{2y} + b_{13j}Z_{1x} + b_{14j}Z_{1y} + b_{15j} \qquad (3)
\end{aligned}$$

where $j = 2,3,4,5,6,7$ and,

$$\begin{aligned}
b_{1j} &= -2(cos\gamma_j - 1) \\
b_{2j} &= -2(cos\psi_j - 1) \\
b_{3j} &= -b_{1j} \\
b_{4j} &= 2sin\gamma_j
\end{aligned}$$

$$
\begin{aligned}
b_{5j} &= -2[(cos\gamma_j - 1)(cos\psi_j - 1) + sin\gamma_j sin\psi_j] \\
b_{6j} &= 2[(cos\gamma_j - 1)sin\psi_j - (cos\psi_j - 1)sin\gamma_j] \\
b_{7j} &= b_{2j} \\
b_{8j} &= 2sin\psi_j \\
b_{9j} &= 2[(cos\gamma_j - 1)\delta_{jx} + sin\gamma_j\delta_{jy}] \\
b_{10j} &= 2[(cos\gamma_j - 1)\delta_{jy} - sin\gamma_j\delta_{jx}] \\
b_{11j} &= 2\delta_{jx} \\
b_{12j} &= 2\delta_{jy} \\
b_{13j} &= -2[(cos\psi_j - 1)\delta_{jx} + sin\psi_j\delta_{jy}] \\
b_{14j} &= 2[sin\psi_j\delta_{jx} - (cos\psi_j - 1)\delta_{jy}] \\
b_{15j} &= \delta_{jx}^2 + \delta_{jy}^2
\end{aligned}
$$

The objective is to solve this system of six equations in six unknowns using continuation. First, however, we must determine the structure of these equations and the number of paths to be tracked.

Number of paths

The number of paths to be tracked depends on the structure of the system of equations to be solved. If the system is m-homogeneous, the number of paths can be reduced. To apply the full continuation method (traditional 1-homogeneous homotopy), the system of equations has to be considered as a 1-homogeneous structure. The number of paths to be tracked for a 1-homogeneous system is equal to the total degree of the equations.

For the case being considered here, the degree of each of the six equations is 2 (refer to Eqs. (3)). Therefore, the total degree of the system is $2^6 = 64$. When expressed in 2- or 3-homogeneous forms the Bezout number of the system is greater than 64. So, the 1-homogeneous form is best for the present problem, and this involves the use of the full continuation method with 64 paths.

Projective transformation

Systems of polynomials with solutions at infinity are common. These solutions result in divergent homotopy paths which have to be identified and aborted. This is a major concern and it would be beneficial if this problem could be avoided. The existence of the solutions at infinity is identified by solving the homogeneous part of Eq. (3), which is obtained by retaining the terms of highest degree (two in the present case) and dropping the remaining terms.

$$
\begin{aligned}
F^{o}_{j-1} = \; & b_{1j}(Z_{3x}^2 + Z_{3y}^2) + b_{2j}(Z_{1x}^2 + Z_{1y}^2) + b_{3j}(Z_{2x}Z_{3x} + Z_{2y}Z_{3y}) \\
& + b_{4j}(Z_{2y}Z_{3x} - Z_{2x}Z_{3y}) + b_{5j}(Z_{1x}Z_{3x} + Z_{1y}Z_{3y}) \\
& + b_{6j}(Z_{1y}Z_{3x} - Z_{1x}Z_{3y}) + b_{7j}(Z_{1x}Z_{2x} + Z_{1y}Z_{2y}) \\
& b_{8j}(Z_{1y}Z_{2x} - Z_{1x}Z_{2y}) \qquad (4)
\end{aligned}
$$

If the solutions at infinity are coefficient independent, the $Z_{1x}, Z_{1y}, Z_{2x}, Z_{2y}, Z_{3x}$ and Z_{3y} values that satisfy Eq. (4) fall into one of the following five categories:

<u>Case 1:</u> $Z_{1x} = Z_{1y} = Z_{3x} = Z_{3y} = 0$, Z_{2x} and Z_{2y} are finite.

<u>Case 2:</u> $Z_{1y} = -iZ_{1x}$, $Z_{2y} = -iZ_{2x}$ and $Z_{3y} = -iZ_{3x}$.

<u>Case 3:</u> $Z_{1y} = iZ_{1x}$, $Z_{2y} = iZ_{2x}$ and $Z_{3y} = iZ_{3x}$.

<u>Case 4:</u> $Z_{2x} = -Z_{1x}$, $Z_{2y} = -Z_{1y}$ and $Z_{3x} = Z_{3y} = 0$.

<u>Case 5:</u> $Z_{1x} = Z_{1y} = 0$, $Z_{3x} = Z_{2x}$ and $Z_{3y} = Z_{2y}$.

It can easily be shown that cases 1, 2 and 3 satisfy Eqs. (4) independent of the coefficients (b_{ij}'s). Cases 4 and 5 can be verified for $b_{7j} = b_{2j}$ and $b_{3j} = -b_{1j}$ respectively. Because the above five groups of solutions are independent of the specified displacements, they will always be present.

Projective transformation is implemented by selecting a homogeneous variable, Z_o, and making the substitutions, $Z_{mn} \leftarrow Z_{mn}/Z_o$ with, $m = 1,2,3$ and $n = x,y$. Substituting Z_{mn} into Eqs. (3) and multiplying by Z_o^2, we get the following homogeneous form:

$$\begin{aligned} F_{j-1}^p = {} & b_{1j}(Z_{3x}^2 + Z_{3y}^2) + b_{2j}(Z_{1x}^2 + Z_{1y}^2) + b_{3j}(Z_{2x}Z_{3x} + Z_{2y}Z_{3y}) \\ & + b_{4j}(Z_{2y}Z_{3x} - Z_{2x}Z_{3y}) + b_{5j}(Z_{1x}Z_{3x} + Z_{1y}Z_{3y}) \\ & + b_{6j}(Z_{1y}Z_{3x} - Z_{1x}Z_{3y}) + b_{7j}(Z_{1x}Z_{2x} + Z_{1y}Z_{2y}) \\ & b_{8j}(Z_{1y}Z_{2x} - Z_{1x}Z_{2y}) + b_{9j}Z_{3x}Z_o + b_{10j}Z_{3y}Z_o \\ & + b_{11j}Z_{2x}Z_o + b_{12j}Z_{2y}Z_o + b_{13j}Z_{1x}Z_o + b_{14j}Z_{1y}Z_o + b_{15j}Z_o^2 \end{aligned} \tag{5}$$

Eqs. (5) represent a system of 6 equations in 7 unknowns. Therefore, the following formula is added which makes Z_o an implicit function of the other variables.

$$\rho_1 Z_{1x} + \rho_2 Z_{1y} + \rho_3 Z_{2x} + \rho_4 Z_{2y} + \rho_5 Z_{3x} + \rho_6 Z_{3y} + \rho_7 Z_o = 1.0 \tag{6}$$

where the ρ_j's are randomly chosen complex constants.

Eq. (6) can be used to eliminate Z_o from Eqs. (5), or Eq. (6) can be solved along with Eqs. (5) (7 equations in 7 unknowns). The latter approach was used to solve the problem at hand. The addition of an extra equation does not affect the number of paths in this case since its degree is one. The computer program, CONSOL 8 [Morgan, 1987], was modified, and the system of equations was solved for seven random complex precision positions. The original variables are then ultimately retrieved from the solutions by use of the transformation, $Z_{mn} = Z_{mn}/Z_o$.

Path tracking

CONSOL 8 uses a predictor-corrector type path tracker. First, the tangent to the continuation curve at the current location is determined and a variable step is taken in this direction. The values are then corrected back to the curve using a Newton's method. Gaussian elimination is used to determine the direction of tangent (for the prediction) and to implement Newton's correction. A detailed description of the path tracking mechanism is provided in Morgan [1987]. Predictor-corrector and other types of path trackers are discussed in Garcia and Zangwill [1981].

Start system

The start system available on CONSOL 8 was used for the full continuation (traditional 1-homogeneous homotopy). This system has the following form:

$$G_k(Z_{mn}) = p_k^{d_k} Z_{mn}^{d_k} - q_k^{d_k} \tag{7}$$

where $k = 1, 2, ..., 6$; $m = 1, 2, 3$ and $n = x, y$. In the above equations, p_k and q_k are independently chosen complex constants, and d_k is the degree of each of the equations to be solved. This start system was used for the initial runs to establish a subsequent start system for the secant homotopy.

Solutions for random displacements

Eqs. (5) and Eq. (6) were solved for a random set of angular displacements, γ_j's and ψ_j's, linear displacements, δ_j's, and ρ_j's. The full continuation was used and 64 paths were tracked. A total of 17 complex solutions and 47 solutions at infinity were obtained for the original system (Eq. (3)), after transformation. Of the solutions at infinity, cases 1, 4 and 5 had three each and cases 2 and 3 had 19 each. The solutions at infinity are all singular and coefficient independent. Therefore, they have no physical meaning. A secant homotopy can be used leaving out the solutions at infinity and tracking just the 17 paths to get all the useful solutions for the system of equations.

Secant homotopy

Solutions at infinity that are not dependent on the coefficients have no physical significance and so they can be avoided. The set of equations solved for the random complex precision points were used as the start system, and the 17 finite solutions were used as the starting points for path tracking using the secant homotopy (refer to Appendix B). This enabled us to get all the useful solutions for the seven position triad synthesis problem by tracking just 17 paths instead of 64. The form of the secant homotopy as presented by Wampler et al. [1990a] is:

$$\mathbf{H}(Zmn, t) = \mathbf{F}^{p}(b_{ij}^{s}, Zmn) \tag{8}$$

where, $m = 1, 2, 3$; $n = x, y$; $i = 1, 2, ..., 15$; $j = 2, 3, ..., 7$ and

$$b_{ij}^{s} = (1 - t)e^{i\theta} b_{ij}(q^{o}) + t b_{ij}(q^{1}) \tag{9}$$

in which, $b_{ij}(q^{o})$ and $b_{ij}(q^{1})$ are the coefficients of Eqs. (5) for randomly chosen displacements and prescribed displacements respectively and θ is a random angle.

Instead of a secant homotopy, a parameter homotopy could have been used. The parameter homotopy is not considered here because it is more complex and might need more CPU time to track the paths.

In order to ensure that the homotopy is working properly and that the system always has the 47 solutions at infinity, problems were solved using both full continuation and the secant homotopy. The solutions at infinity were observed to have the same pattern as that indicated in the "solutions for Random Displacement" section. The secant homotopy gave all the useful solutions for the problems by tracking just 17 paths.

The computation time required to solve one particular complex precision point set was 17.4 minutes for full continuation and 5.6 minutes for secant homotopy. For a real set of precision points, it was 24.3 minutes and 9.7 minutes respectively for the full and secant homotopies. Though time taken per path is more for a secant homotopy, the overall time to solve a problem is far less than that of full continuation due to the elimination of extraneous solutions at infinity.

Examples

Synthesis of triads satisfying the two sets of precision points listed in Table 1 was carried out using the secant homotopy. The real solutions obtained are given in Table 2.

Example 1 resulted in five real solutions, eight complex solutions and four solutions at infinity. It took 11.5 minutes of computation time to solve this problem. Example 2 took 8.1 minutes of CPU time. A total of six real solutions, eight complex solutions and 3 solutions at infinity were obtained. Solution 4 of example 2 is shown in the seven prescribed positions in Fig. 2. All computations were carried out on a VAX 11/785 computer system.

Table 1: Precision points for the Triad synthesis

Precision Point	δ_{jx}	δ_{jy}	γ_j deg.	ψ_j deg.
		Example 1		
2	0.5	0.0	10.0	5.0
3	1.0	0.0	20.0	10.0
4	1.5	0.0	30.0	15.0
5	2.0	0.0	40.0	20.0
6	2.5	0.0	50.0	25.0
7	3.0	0.0	60.0	30.0
		Example 2		
2	1.5	0.5	30.0	15.0
3	2.5	1.0	60.0	30.0
4	3.0	1.6	90.0	45.0
5	3.0	2.0	120.0	60.0
6	3.5	2.5	150.0	75.0
7	4.0	3.0	180.0	90.0

Table 2: Triads satisfying the prescribed conditions

No.	Z_{1x}	Z_{1y}	Z_{2x}	Z_{2y}	Z_{3x}	Z_{3y}
			Example 1			
1	-4.62178	2.69312	11.3495	-18.4818	0.09196	0.29848
2	-40.6734	-128.999	48.9816	122.474	10.0583	14.3647
3	-29.2756	-132.054	35.4366	127.056	7.41106	15.8931
4	-35.1309	-131.110	42.4036	125.393	8.77266	15.1947
5	5.34914	0.02142	-14.7724	-15.8797	0.21251	0.22888
			Example 2			
1	-3.46580	-4.80846	0.55267	-0.01185	-2.13588	0.84965
2	-1.85382	-5.01414	-0.31039	0.41801	-1.44449	-0.08757
3	13.6559	12.8170	-40.8693	-10.3366	-25.3305	2.15534
4	-5.37799	-2.13705	1.99701	-0.45720	-0.32524	1.96391
5	-0.61860	-6.34327	-0.28397	1.04024	-1.09445	-0.94899
6	-2.17396	-5.27678	0.00640	0.33788	-1.86942	0.200383

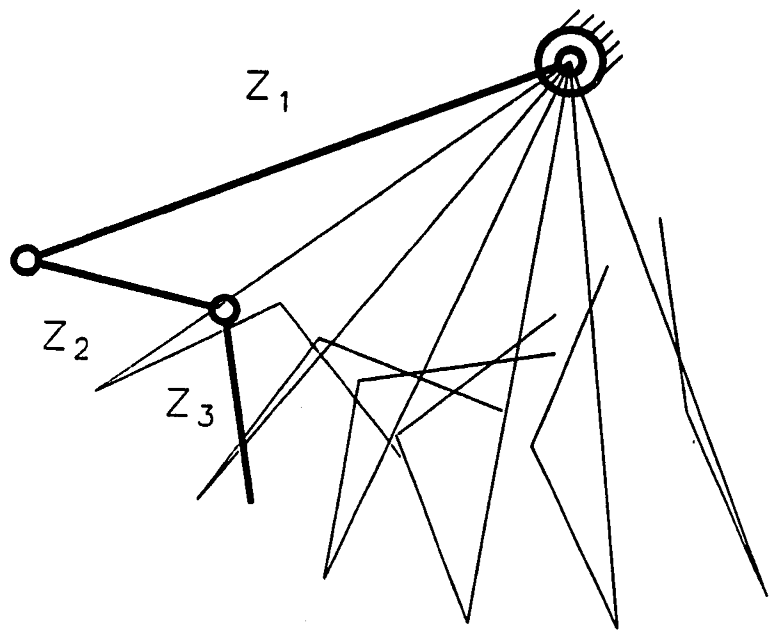

Figure 2: Triad 4 of example 2 in the seven prescribed positions

REVOLUTE-SPHERICAL (R-S) DYAD SYNTHESIS

Spatial dyad synthesis is useful for motion generation applications. As in planar mechanism synthesis, single and multi-loop linkages can be broken up into combinations of dyads and the components can be designed independently. Synthesis of spatial dyads has been considered by Roth [1968], Tsai and Roth [1971], Suh and Radcliffe [1978], and Sandor et al. [1984] to name a few. Design of an R-S dyad, shown in Fig. 3, for four position motion generation is considered here. This dyad is a part of a variety of multi-loop mechanisms such as the RSSR-SS and RSSR-RC. Development of the equations and the procedure needed to implement the parameter homotopy to solve the problem are presented in the following sections.

Development of equations

An R-S dyad in two finitely separated positions is shown in Fig. 3. The constraint equations for the link displacement [Suh and Radcliffe, 1978] may be expressed as:

$$u_{ox}(a_{kx} - a_{ox}) + u_{oy}(a_{ky} - a_{oy}) + u_{oz}(a_{kz} - a_{oz}) = 0 \tag{10}$$

$$u_{ox}^2 + u_{oy}^2 + u_{oz}^2 - 1 = 0 \tag{11}$$

$$(a_{jx} - a_{ox})^2 + (a_{jy} - a_{oy})^2 + (a_{jz} - a_{oz})^2 = (a_{1x} - a_{ox})^2 + (a_{1y} - a_{oy})^2 + (a_{1z} - a_{oz})^2 \tag{12}$$

where, $k = 1, 2, 3, 4$, $j = 2, 3, 4$ and $\mathbf{a}_j = [D_{1j}]\mathbf{a}_1$.

In the above expression, $[D_{1j}]$ is the displacement matrix from position 1 to j which is specified for a motion generation problem. Since the displacement equations are eight, one of the unknowns should be assigned an arbitrary value. The value of

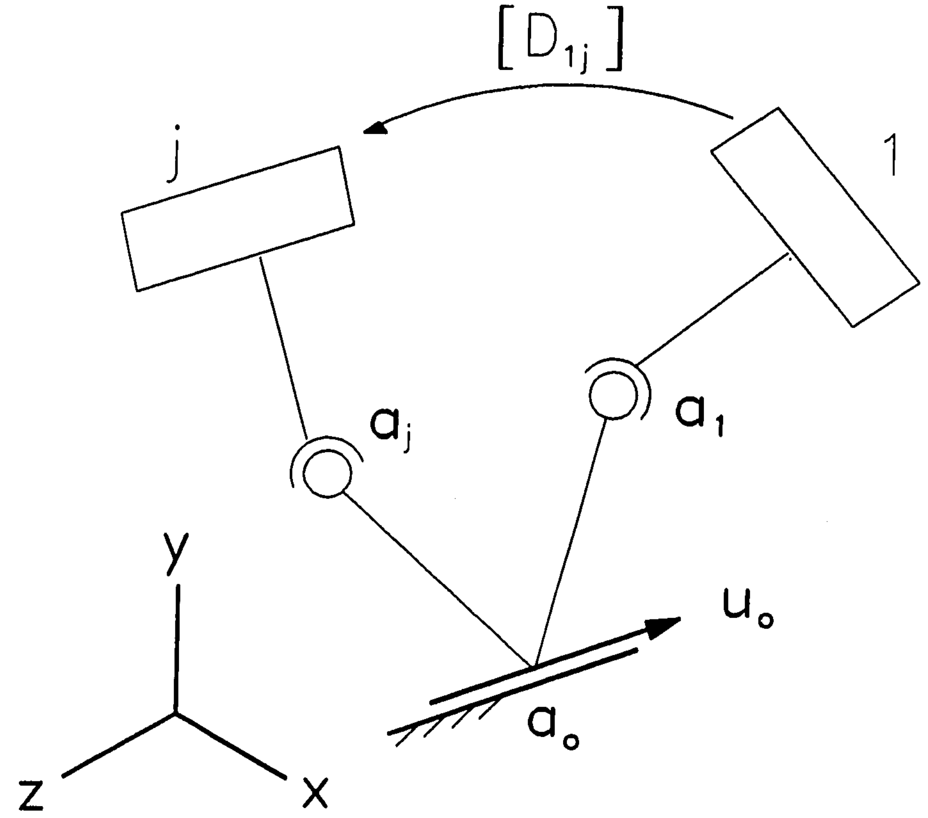

Figure 3: A dyad in two finitely displaced positions

u_{ox} is fixed to be between -1.0 and 1.0 and the unknowns a_{1x}, a_{1y}, a_{1z}, a_{ox}, a_{oy}, a_{oz}, u_{oy} and u_{oz} are determined. On rewriting the equations using x_j, $j = 1, 2, ..., 8$ for the original unknowns and after substituting for $\mathbf{a}_j$, we obtain the following equations:

$$\begin{aligned} F_k &= B_{1k}x_1x_7 + B_{2k}x_1x_8 + B_{3k}x_2x_7 + B_{4k}x_2x_8 + B_{5k}x_3x_7 + \\ &\quad B_{6k}x_3x_8 + B_{7k}x_5x_7 + B_{8k}x_6x_8 + B_{9k}x_1 + B_{10k}x_2 + B_{11k}x_3 + \\ &\quad B_{12k}x_4 + B_{13k}x_7 + B_{14k}x_8 + B_{15k} = 0 \qquad (13) \\ F_5 &= x_7^2 + x_8^2 + (u_{ox}^2 - 1.0) = 0 \qquad (14) \\ F_{j+4} &= A_{1j}x_1x_2 + A_{2j}x_1x_3 + A_{3j}x_2x_3 + A_{4j}x_1x_4 + A_{5j}x_1x_5 + \\ &\quad A_{6j}x_1x_6 + A_{7j}x_2x_4 + A_{8j}x_2x_5 + A_{9j}x_2x_6 + A_{10j}x_3x_4 + \\ &\quad A_{11j}x_3x_5 + A_{12j}x_3x_6 + A_{13j}x_1 + A_{14j}x_2 + A_{15j}x_3 + A_{16j}x_4 + \\ &\quad A_{17j}x_5 + A_{18j}x_6 + A_{19j} = 0 \qquad (15) \end{aligned}$$

where B_{ik}'s and A_{mj}'s ($i = 1, 2, ..., 15$ and $m = 1, 2, ..., 19$) are constants determined from the specified displacement matrices and the arbitrary choice for u_{ox}. In order to solve this system of equations, we need to first determine the structure and the number of paths to be tracked.

Number of paths

The degree of each of the eight equations to be solved is two, and the total degree of the system is $2^8 = 256$. Therefore, a 1-homogeneous homotopy would require 256 paths to determine the complete solution set for the system.

The design equations are of 2-homogeneous structure which can be identified by grouping the variables into two sets as follows:

Set 1: $x_1, x_2, x_3, x_4, x_5\ x_6$

Set 2: x_7, x_8

The degree of the variables in set 1 is one in the first four equations (Eqs. (13)), zero in the fifth (Eq. (14)) and two in the final three (Eqs. (15)) whereas the degree for set 2 is one, two and zero respectively. If α_1 represents set 1 and α_2 set 2, the Bezout number (refer to Appendix A) is the coefficient of $\alpha_1^6\alpha_2^2$ in the product $(\alpha_1+\alpha_2)^4(2\alpha_2)(2\alpha_1)^3$. This coefficient turns out to be 64. Therefore, since the system is 2-homogeneous, the number of paths that need to be tracked to determine all the solutions can be reduced to 64. A detailed description of the procedure to determine the Bezout number is given in Morgan and Sommese [1987a, 1987b].

Projective transformation

Since the 2-homogeneous structure of the equations is to be utilized, we have to use two homogeneous variables. If x_{o1} and x_{o2} are the homogeneous variables corresponding to sets 1 and 2, we homogenize the equations by making the substitution, $x_j \leftarrow x_j/x_{o1}$ for $j = 1,2,3,4,5,6$ and $x_j \leftarrow x_j/x_{o2}$ for $j = 7,8$. On substituting and modifying we get the following expressions:

$$\begin{aligned} F_k^p &= B_{1k}x_1x_7 + B_{2k}x_1x_8 + B_{3k}x_2x_7 + B_{4k}x_2x_8 + B_{5k}x_3x_7 + B_{6k}x_3x_8 + \\ &\quad B_{7k}x_5x_7 + B_{8k}x_6x_8 + B_{9k}x_1x_{o2} + B_{10k}x_2x_{o2} + B_{11k}x_3x_{o2} + \\ &\quad B_{12k}x_4x_{o2} + B_{13k}x_7x_{o1} + B_{14k}x_8x_{o1} + B_{15k}x_{o1}x_{o2} = 0 \qquad (16) \\ F_5^p &= x_7^2 + x_8^2 + (u_{ox}^2 - 1.0)x_{o2}^2 = 0 \qquad (17) \end{aligned}$$

$$F^p_{j+4} = A_{1j}x_1x_2 + A_{2j}x_1x_3 + A_{3j}x_2x_3 + A_{4j}x_1x_4 + A_{5j}x_1x_5 + A_{6j}x_1x_6 + A_{7j}x_2x_4 + A_{8j}x_2x_5 + A_{9j}x_2x_6 + A_{10j}x_3x_4 + A_{11j}x_3x_5 + A_{12j}x_3x_6 + A_{13j}x_1x_{o1} + A_{14j}x_2x_{o1} + A_{15j}x_3x_{o1} + A_{16j}x_4x_{o1} + A_{17j}x_5x_{o1} + A_{18j}x_6x_{o1} + A_{19j}x_{o1}^2 = 0 \quad (18)$$

Since the number of equations to be solved are 8 involving 10 unknowns, two additional equations are added to the system. These additional formulae are chosen to be of degree one and each equation should be an implicit function of the variables pertaining to one set only.

$$F^p_9 = \rho_1x_1 + \rho_2x_2 + \rho_3x_3 + \rho_4x_4 + \rho_5x_5 + \rho_6x_6 + \rho_7x_{o1} - 1.0 = 0 \quad (19)$$

$$F^p_{10} = \rho_8x_7 + \rho_9x_8 + \rho_{10}x_{o2} - 1.0 = 0 \quad (20)$$

Since the degree of Eq. (19) and Eq. (20) are chosen to be one, they will not affect the number of paths to be tracked. Now, by solving the above 10 equations in 10 unknowns, we avoid the solutions at infinity. The transformation, $x_j = x_j/x_{o1}$ $j = 1,2,3,4,5,6$ and $x_j = x_j/x_{o2}$ $j = 7,8$, should be performed to obtain the solutions of the original system (Eqs. (13), (14) and (15)).

Start system

The start system should have the same 2-homogeneous structure as the target system. The following easy to solve start system was used to determine the solutions of the target system:

$$G_1 = x_1^2 - 4 = 0$$

$$G_2 = x_2^2 - 36 = 0$$

$$
\begin{aligned}
G_3 &= x_3^2 - 9 = 0 \\
G_4 &= (x_1 + 2x_4 + 3x_5 + 4x_6 - 2)(x_7 + x_8 - 4) = 0 \qquad (21) \\
G_5 &= (x_2 + x_4 + 2x_5 + x_6 - 3)(x_7 + x_8 - 5) = 0 \\
G_6 &= (x_3 + 3x_4 + x_5 + 2x_6 - 4)(x_7 + x_8 - 6) = 0 \\
G_7 &= (x_1 + x_4 + x_5 + 3x_6 - 5)(x_7 + x_8 - 7) = 0 \\
G_8 &= x_8^2 - 81 = 0
\end{aligned}
$$

Note that the Bezout number of the start system is the same as that of the target system. This 2-homogeneous start system can be combined with Eqs. (19) and (20) to obtain the final start system to solve Eqs. (16), (17) and (18).

Solutions for random displacements

In order to eliminate the extraneous solutions at infinity, and to implement a parameter homotopy, the 2-homogeneous start system was used to solve Eqs. (16), (17) and (18) for a random set of displacement matrices, $[D_{ij}]$'s, and u_{ox} value. The predictor-corrector type path tracker available on CONSOL 8 was used after modifying the program to incorporate the new start system. The procedure resulted in 22 complex solutions and 42 solutions at infinity. As for the triad synthesis, the solutions at infinity are coefficient independent and so have no physical meaning. It took about 50 minutes of computation time to track the 64 paths. Different sets of complex precision points were used and the solutions were evaluated to make sure that none of the finite solutions were missed.

Parameter homotopy

The next step requires use of the system of equations solved for random complex precision positions as the start system along with a secant or parameter homotopy (refer to Appendix B) to eliminate the 42 solutions at infinity. A secant homotopy did not give proper convergence and so a parameter homotopy was implemented. The parameter homotopy is given by [Wampler et al., 1990a]:

$$\mathbf{H}(x_j, t) = \mathbf{F}^p(B^p_{ik}, A^p_{mj}, u^p_{ox}, x_j) \tag{22}$$

where,

$$\begin{aligned} B^p_{ik} &= B_{ik}[(1-t)q^o + tq^1] \\ A^p_{mj} &= A_{mj}[(1-t)q^o + tq^1] \\ u^p_{ox} &= (1-t)u^o_{ox} + tu^1_{ox} \end{aligned}$$

where q^o's and u^o_{ox} are the randomly chosen values and q^1's are the prescribed precision points. The u^1_{ox} value is selected arbitrarily between -1 and 1.

A number of sample problems were solved to ensure that the parameter homotopy would give all the solutions for the specified precision points by tracking 22 paths. Two examples are presented in the following section.

Examples

The precision points given in Table 3 were used and R-S dyads satisfying them were determined using a parameter homotopy. The real solutions obtained are listed in Table 4.

Example 1 is from Sandor et al. [1986]. They set up error functions and minimized them using the Hooke Jeeves method. Here precision position synthesis is

Table 3: Precision points for the design of R-S dyad

Example 1

$$u_{ox} = 0.431$$

$$[D_{12}] = \begin{bmatrix} 0.837 & -0.224 & 0.500 & -0.504 \\ 0.408 & 0.863 & -0.296 & 2.262 \\ -0.365 & 0.452 & 0.814 & 1.050 \end{bmatrix}$$

$$[D_{13}] = \begin{bmatrix} 0.694 & -0.324 & 0.643 & -0.333 \\ 0.657 & 0.649 & -0.383 & 4.159 \\ -0.293 & 0.688 & 0.633 & 2.613 \end{bmatrix}$$

$$[D_{14}] = \begin{bmatrix} 0.433 & -0.250 & 0.866 & 0.706 \\ 0.896 & 0.225 & -0.383 & 8.747 \\ -0.099 & 0.942 & 0.321 & 5.497 \end{bmatrix}$$

Example 2

$$u_{ox} = 0.7072135$$

$$[D_{12}] = \begin{bmatrix} 0.943 & 0.328 & 0.056 & -0.625 \\ -0.331 & 0.942 & 0.063 & 0.704 \\ -0.032 & -0.078 & 0.996 & -0.169 \end{bmatrix}$$

$$[D_{13}] = \begin{bmatrix} 0.535 & 0.838 & 0.105 & -0.623 \\ -0.823 & 0.490 & 0.285 & 2.624 \\ 0.188 & -0.239 & 0.953 & -1.296 \end{bmatrix}$$

$$[D_{14}] = \begin{bmatrix} 0.233 & 0.948 & 0.218 & 0.334 \\ -0.831 & 0.077 & 0.551 & 3.024 \\ 0.505 & -0.309 & 0.805 & -2.739 \end{bmatrix}$$

Table 4: Dyads satisfying the prescribed conditions

No.	a_{1x}	a_{1y}	a_{1z}	a_{ox}	a_{oy}	a_{oz}	u_{oy}	u_{oz}
				Example 1				
1	-8.904	4.440	-26.37	-0.219	-0.961	1.608	0.901	0.042
2	-4.006	-6.380	-0.571	-13.01	2.976	5.886	-0.193	0.881
3	78.99	91.62	87.10	83.40	79.18	99.39	0.707	0.561
4	-2.193	-2.378	0.838	-24.17	14.36	-7.124	0.142	-0.891
5	-2.885	-1.098	5.904	-0.013	5.192	6.354	-0.206	-0.879
6	11.94	9.183	17.92	15.16	12.34	19.04	-0.655	0.620
7	5.729	2.303	14.92	9.294	8.247	14.86	-0.250	0.867
				Example 2				
1	5.006	6.006	-5.014	-0.002	9.401	-0.004	0.0	0.707
2	-5.178	-5.631	20.25	7.823	1.208	5.857	0.121	0.697
3	0.943	3.533	0.929	1.520	-0.660	-1.170	-0.236	0.666
4	3.084	4.250	-1.494	-0.158	-1.377	1.088	-0.085	0.702

being applied to solve the same problem. Eight real solutions and 14 complex solutions were obtained using 13.1 minutes of computation time. Only 7 real solutions are listed in Table 4, the other solution resulted in very large link lengths. Example 2 resulted in 6 real solutions and 16 complex solutions. The computation time involved was 13.7 minutes. Two of the real solutions have very high magnitudes and so are not included in the table. All computations were carried out on a VAX 11/785 machine as before.

REFERENCES

Chase, T. R., A. G. Erdman, and D. R. Riley. 1987. Triad Synthesis for up to Five Design Positions With Application to the Design of Arbitrary Planar Mechanisms. Journal of Mechanisms, Transmissions, and Automation in Design 109:426-434.

Garcia, C. B. and W. I. Zangwill. 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, New Jersey.

Lin, C. -S., and A. G. Erdman. 1987. Dimensional Synthesis of Planar Triads: Motion Generation with Prescribed Timing for Six Precision Positions. Journal of Mechanism and Machine Theory 22:411-419.

Morgan, A. P. 1986a. A Homotopy for Solving Polynomial Systems. Applied Mathematics and Computation 18:87-92.

Morgan, A. P. 1986b. A Transformation to Avoid Solutions at Infinity for Polynomial Systems. Applied Mathematics and Computation 18:77-86.

Morgan, A. P. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Morgan, A. P., and A. J. Sommese. 1987a. Computing All Solutions to Polynomial Systems Using Homotopy Continuation. Applied Mathematics and Computation 24:115-138.

Morgan, A. P., and A. J. Sommese. 1987b. A Homotopy for Solving General Polynomial Systems That Respects m-Homogeneous Structures. Applied Mathematics and Computation 24:101-113.

Morgan, A. P., and A. J. Sommese. 1989. Coefficient-Parameter Polynomial Continuation. Applied Mathematics and Computation 29:123-160.

Morgan, A. P., and C. W. Wampler. 1989. Solving a Planar Four-Bar design Problem Using Continuation. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:409-416.

Roth, B. 1968. The Design of Binary Cranks with Revolute, Cylindric, and Prismatic Joints. Journal of Mechanisms 3:61-72.

Roth, B., and F. Freudenstein. 1963. Synthesis of Path-Generating Mechanisms by Numerical Methods. ASME Journal of Engineering for Industry 85B, no.3:298-306.

Sandor, G. N., D. Kohli, C. F. Reinholtz, and A. Ghosal. 1984. Closed-Form Analytic Synthesis of a Five-Link Spatial Motion Generator. Mechanism and Machine Theory 19, no.1:97-105.

Sandor, G. N., L. J. XU, and S. P. Yang. 1986. Computer-Aided Synthesis of Two-Closed-Loop RSSR-SS Spatial Motion Generator With Branching and Sequence Constraints. Mechanism and Machine Theory 21, no.4:345-350.

Starns, K. G., and D. R. Flugrad. 1990. Five-Bar Path Generation Synthesis by a Continuation Method. Mechanism Synthesis and Analysis. ASME DE-25:353-359.

Subbian, T., and D. R. Flugrad. 1989a. Synthesis of Motion Generating Mechanisms by a Continuation Method. Proceedings of the 1st National Applied Mechanisms and Robotics Conference 89AMR-2B-2.

Subbian, T., and D. R. Flugrad. 1989b. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:425-432.

Subbian, T., and D. R. Flugrad. 1990. Five Position Triad Synthesis With Applications to Four- and Six-Bar Mechanisms. To appear in the Journal of Mechanical Design.

Suh, C. H., and C. W. Radcliffe. 1978. Kinematics and Mechanism Design. John Wiley and Sons, Inc., New York.

Tsai, L. W., and J. J. Lu. 1989. Coupler-Point-Curve Synthesis Using Homotopy methods. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:417-424.

Tsai, L. W., and B. Roth. 1971. Design of Dyads with Helical, Cylindrical, Spherical, Revolute and Prismatic Joints. Mechanism and Machine Theory 7:85-102.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990a. Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics. Journal of Mechanical Design 112:59-68.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990b. Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages. Mechanism Synthesis and Analysis. ASME DE-25:361-368.

PART VI.

ROBOT TRAJECTORY PLANNING BY A CONTINUATION METHOD

INTRODUCTION

Closed-form inverse kinematic solutions for a general robot are difficult to obtain. Trajectory planning for manipulators is even more complicated and has been the focus of many researchers. Some investigators have converted the cartesian path tracking problem to that of a joint space problem for convenience. Taylor [1979], Goldenberg and Lawrence [1986], Hornick and Ravani [1986], Wang [1988], and Chand and Doty [1985] have obtained the inverse kinematic solutions for a finite number of knot points on the specified trajectory and have worked in joint space to achieve the desired motion.

Evaluation of the inverse kinematic solutions for an end effector to trace a specified continuous path is extremely important for applications such as welding and painting. Some researchers have developed iterative schemes to evaluate inverse kinematic solutions continuously along trajectories. Gupta and Kazerounian [1985] conducted a comparative study between modified Newton-Raphson and modified predictor-corrector methods and showed that the predictor-corrector method is better. Singh and Gupta [1989] implemented a modified Jacobian based Newton-Raphson scheme to trace a trajectory. Both the methods calculate the joint velocities and use iterative techniques to determine the angular displacements. Angeles [1986] used a continuation method to move to the starting point of the trajectory and later used a least squares approximation to determine the joint angles, velocity and acceleration histories along the path. The major problem in implementing numerical methods to solve path trajectory problems has been the computation time required. The number of intermediate points at which inverse kinematic solutions can be evaluated depends on the computation time available for an on-line application.

A continuation method approach is proposed in here which reduces the path generating problem to that of solving a system of first order ordinary differential equations for the joint angles with respect to the cartesian path variable. This opens up the possibility of developing this method for on-line trajectory planning. Since we are integrating for the joint variables with respect to a path variable, we can also obtain joint velocity information. Furthermore, in this procedure, position and velocity information are available at each instant of time and can be used for simulating the robot off-line. In fact, continuation methods could be used to develop off-line trajectory planning packages similar to those described by Hornick and Ravani [1986] and Pickett and Zarger [1983].

The basic concept behind the method here is similar to that of Gupta and his associates. However, in establishing a continuation formulation for the problem, a path parameter is used as the independent variable rather than time.

Application of continuation methods to solve a 3-revolute (3-R) robot is considered in detail. The procedure is used in particular to determine the inverse kinematic solutions for tracing a circular path for two different configurations of the 3-R robot. The procedure is then extended to a six degree of freedom PUMA robot, again to track a circular path.

CONTINUATION METHOD

Continuation methods in the last few years have received considerable attention in applications to kinematic analysis and synthesis problems. Subbian and Flugrad [1989a, 1989b, 1990], Starns and Flugrad [1990], Morgan and Wampler [1989], Wampler et al. [1990a] and Tsai and Lu [1989] applied this procedure to synthesize mechanisms. Jo and Haug [1988] carried out the work space analysis of two degree-of-freedom robot arms and slider-crank mechanisms utilizing continuation methods. Also, Tsai and Morgan [1985] and Wampler and Morgan [1989] succeeded in determining the inverse kinematic solutions for five and six degree-of-freedom manipulators using this technique. Here, we are applying continuation methods to solve trajectory planning problems.

Continuation methods can be used to solve a system of n equations in either n or $(n+1)$ unknowns. To solve a system of n equations in n unknowns, say $\mathbf{F}(z) = 0$, a simple system is assumed to be a start system, say $\mathbf{G}(z) = 0$. This start system must be of the same degree as the original system and easy to solve. Homotopy functions can then be written combining the two systems of equations as $\mathbf{H}(z,t) = \mathbf{F}(z)t + \mathbf{G}(z)(1-t) = 0$, where t is the homotopy parameter and $\mathbf{H}$ the homotopy function. When $t = 0$, the homotopy function reduces to the start system ; when $t = 1$ it becomes the original system of equations to be solved. Therefore, by increasing t from 0 to 1 and simultaneously tracking the values of the z variables, the original system $\mathbf{F}(z) = 0$ is solved. The z variables are tracked by integrating a set of first order ordinary differential equations with respect to the homotopy parameter t. The solutions for the start system are used as initial conditions for the integration.

When solving a system of n equations in $(n+1)$ unknowns, solution curves are obtained, rather than a finite solution set. The procedure involved is a two step process. First, at least one point on each of the solution curves is determined. Suppose $\mathbf{E}(x) = 0$ is the system of n equations in $(n+1)$ unknowns. The procedure described by Morgan [1981] utilizes an extended Jacobian matrix of $\mathbf{E}(x) = 0$ to determine an $(n+1)$th equation. Hence, a set of $(n+1)$ equations in $(n+1)$ unknowns is solved using continuation to find the finite solution set of interest. Obtaining the $(n+1)$th equation analytically, however, can be difficult for a complicated system of equations; therefore an alternative approach is used here.

The system of equations under consideration has a trajectory path variable as the $(n+1)$th variable, which is 0 at the initial point and 1 at the terminal point. The path traversed by the robot between these two points is specified. The remaining n variables are the joint angles, whose values are to track the prescribed path. By setting $x_{n+1} = 0$ then, we can determine the inverse kinematic solutions for the robot at the initial point. These inverse kinematic solutions can either be calculated by closed-form solutions, or by using a continuation method [Tsai and Morgan, 1985, Wampler and Morgan, 1989].

The finite solutions obtained by the above procedures give points on the solution curves. The second step would be to trace the solution curves from the initial points, obtained above, by integrating a set of n first order ordinary differential equations of the n variables with respect to x_{n+1}. The differential equations are obtained from the extended Jacobian matrix of the given system of equations ($\mathbf{E}(x) = 0$). A detailed description of the procedures outlined here is provided in Subbian and Flugrad [1989a, 1989b], Morgan [1981, 1987] and Wampler et al. [1990b].

DEVELOPMENT OF EQUATIONS FOR THE 3-R MANIPULATOR

The 3-R manipulator for the two configurations considered is shown in Fig. 1. Shown in Fig. 2 is a generalized link $(i-1)$ paired at a joint i with another link, link (i). Corresponding cartesian frames are attached to the links with axes Z_{i-1} and Z_i, alligned with the joint axes $(i-1)$ and (i), respectively. The link parameters as used by Craig [1986] are shown, where the angle θ_i is the position variable at joint (i) for a revolute joint. The parameters for the configurations 1 and 2 (in Fig. 1) are listed in Table 1.

Table 1: Link parameters for the two configurations

Configuration 1

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-90^o	0	0	θ_2
3	0	17.0	4.9	θ_3
4	-90^o	0.8	17.0	0

Configuration 2

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	-70^o	0.5	2.0	θ_2
3	30^o	17.0	4.9	θ_3
4	-90^o	0.8	17.0	0

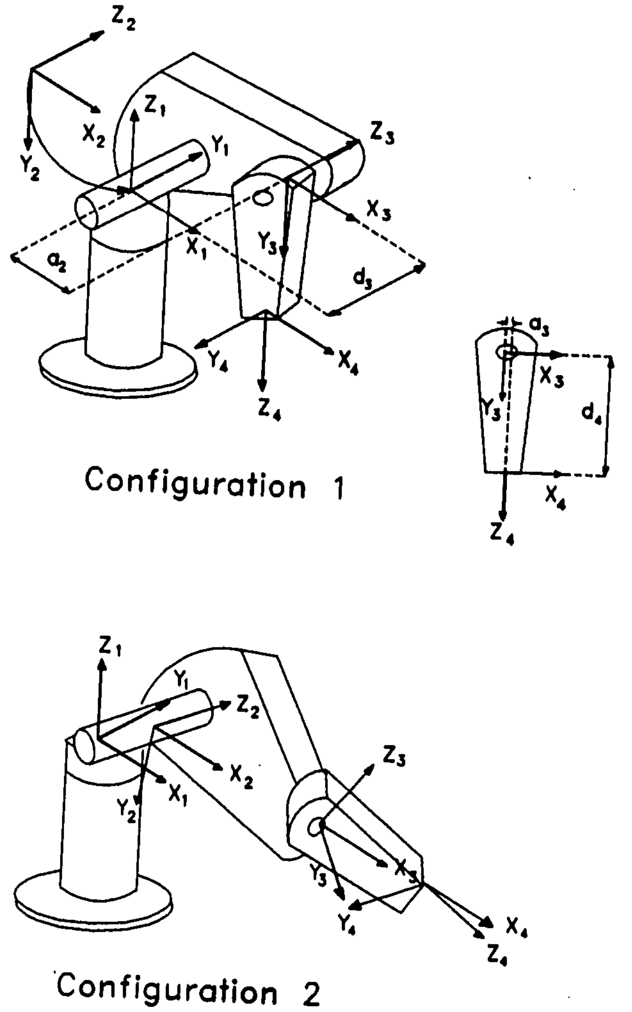

Figure 1: The 3-R manipulator

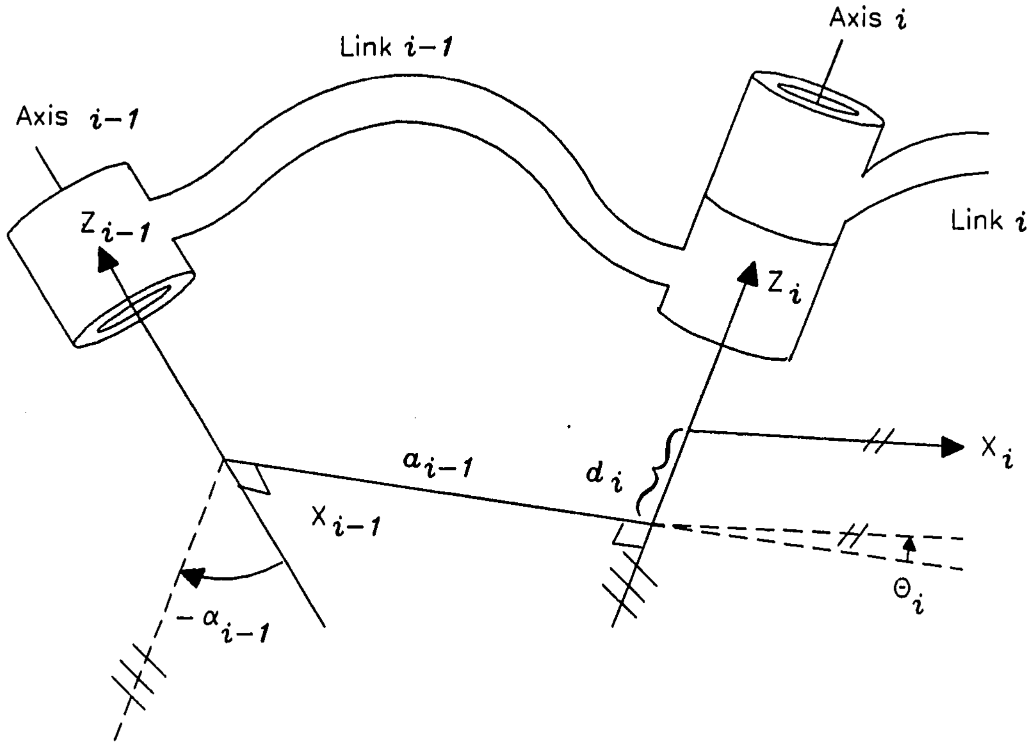

Figure 2: Definition of link parameters

The homogeneous transformation matrix relating frames $\{i\}$ and $\{i-1\}$ is of the form:

$$
{}^{i-1}_{i}T = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ S\theta_i C\alpha_{i-1} & C\theta_i C\alpha_{i-1} & -S\alpha_{i-1} & -S\alpha_{i-1}d_i \\ S\theta_i S\alpha_{i-1} & C\theta_i S\alpha_{i-1} & C\alpha_{i-1} & C\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}
$$

Where the notations C and S indicate *cos* and *sin*, respectively. A fourth transformation matrix is written for the tool frame $\{4\}$ coordinate system fixed to link 3:

$$
{}^{3}_{4}T = \begin{bmatrix} 1 & 0 & 0 & 0.8 \\ 0 & 0 & 1 & 17.0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}
$$

The kinematics of the manipulator, for either configuration 1 or 2, are expressed then by evaluating Eq. (1) for i =1, 2 and 3 and by combining the results with Eq. (2) to find ${}^{0}_{4}T = {}^{0}_{1}T\,{}^{1}_{2}T\,{}^{2}_{3}T\,{}^{3}_{4}T$.

The position of frame $\{4\}$ in the cartesian space, frame $\{0\}$, found in column 4 of ${}^{0}_{4}T$ for configuration 1 is given by:

$$
\begin{aligned}
P_x &= 0.8(C\theta_1 C\theta_2 C\theta_3 - C\theta_1 S\theta_2 S\theta_3) - 17.0(C\theta_1 C\theta_2 S\theta_3 + C\theta_1 S\theta_2 C\theta_3) + \\
&\quad 17.0 C\theta_1 C\theta_2 - 4.9 S\theta_1 \\
P_y &= 0.8(S\theta_1 C\theta_2 C\theta_3 - S\theta_1 S\theta_2 S\theta_3) - 17.0(S\theta_1 C\theta_2 S\theta_3 + S\theta_1 S\theta_2 C\theta_3) + \\
&\quad 17.0 S\theta_1 C\theta_2 + 4.9 C\theta_1 \qquad (3) \\
P_z &= -0.8(S\theta_2 C\theta_3 + C\theta_2 S\theta_3) - 17.0(C\theta_2 C\theta_3 - S\theta_2 S\theta_3) - 17.0 S\theta_2
\end{aligned}
$$

Similarly, for the more general configuration 2:

$$
\begin{aligned}
P_x &= 0.8(C\theta_1 C\theta_2 C\theta_3 - 0.342 S\theta_1 S\theta_2 C\theta_3 - 0.866 C\theta_1 S\theta_2 S\theta_3 \\
&\quad -0.2962 S\theta_1 C\theta_2 S\theta_3 - 0.4698 S\theta_1 S\theta_3) + \\
&\quad 17.0(-C\theta_1 C\theta_2 S\theta_3 + 0.342 S\theta_1 S\theta_2 S\theta_3 - 0.866 C\theta_1 S\theta_2 C\theta_3 \\
&\quad -0.2962 S\theta_1 C\theta_2 C\theta_3 - 0.4698 S\theta_1 C\theta_3) + \\
&\quad 17.0 C\theta_1 C\theta_2 + 2.45 C\theta_1 S\theta_2 - 5.8143 S\theta_1 S\theta_2 + 0.8379 S\theta_1 C\theta_2 \\
&\quad +2.1082 S\theta_1 + 0.5 C\theta_1 \\
P_y &= 0.8(S\theta_1 C\theta_2 C\theta_3 + 0.342 C\theta_1 S\theta_2 C\theta_3 - 0.866 S\theta_1 S\theta_2 S\theta_3 \\
&\quad +0.2962 C\theta_1 C\theta_2 S\theta_3 + 0.4698 C\theta_1 S\theta_3) + \\
&\quad 17.0(-S\theta_1 C\theta_2 S\theta_3 - 0.342 C\theta_1 S\theta_2 S\theta_3 - 0.866 S\theta_1 S\theta_2 C\theta_3 \\
&\quad +0.2962 C\theta_1 C\theta_2 C\theta_3 + 0.4698 C\theta_1 C\theta_3) + \\
&\quad 17.0 S\theta_1 C\theta_2 + 2.45 S\theta_1 S\theta_2 + 5.8143 C\theta_1 S\theta_2 - 0.8379 C\theta_1 C\theta_2 \\
&\quad +5.876 C\theta_1 + 0.5 S\theta_1 \qquad (4) \\
P_z &= -0.7518 S\theta_2 C\theta_3 - 0.651 C\theta_2 S\theta_3 + 0.1368 S\theta_3 + \\
&\quad +15.9748 S\theta_2 S\theta_3 - 13.8346 C\theta_2 C\theta_3 + 2.9072 C\theta_3 \\
&\quad -15.9748 S\theta_2 + 2.3022 C\theta_2 + 2.1354
\end{aligned}
$$

INVERSE KINEMATIC SOLUTION AT THE START OF THE TRAJECTORY

For configuration 1 of the 3-R robot, closed-form solutions of the inverse kinematics can be determined from Eqs. (3). On reduction, the following relationships relating the joint and cartesian space variables can be obtained [Craig, 1986],

$$\begin{aligned}
\theta_1 &= Atan2(P_y, P_x) - Atan2(4.9, +\sqrt{P_x^2 + P_y^2 - 24.01}) \\
\theta_3 &= Atan2(0.8, 17.0) - Atan2(K, +\sqrt{289.64 - K^2}) \\
where, K &= (P_x^2 + P_y^2 + P_z^2 - 602.65)/34.0 \\
\theta_2 &= Atan2((-0.8 - 17.0C\theta_3)P_z - (C\theta_1 P_x + S\theta_1 P_y)(17.0 - 17.0S\theta_3), \\
&\quad (17.0S\theta_3 - 17.0)P_z + (0.8 + 17.0C\theta_3)(C\theta_1 P_x + S\theta_1 P_y)) - \theta_3
\end{aligned} \tag{5}$$

Inverse kinematic solutions were determined using these relationships for the manipulator at the start position (22, 5, 10) in cartesian space. Table 2 provides the four inverse kinematic solutions obtained.

For configuration 2, it is difficult to determine a closed-form inverse kinematics solution. Therefore, some sort of numerical technique is needed. The continuation methods used by Tsai and Morgan [1985] as well as Wampler and Morgan [1989] are difficult to implement. Therefore, an alternative approach is proposed which takes advantage of the fact that the set of equations to be solved (Eqs. (4)) is similar to the set for configuration 1 (Eqs. (3)). The homotopy function used has the following form:

$$\mathbf{H}(y, t) = (1 - t)e^{i\theta}\mathbf{G}(y) + t\mathbf{F}(y) \tag{6}$$

where $\mathbf{G}(y) = 0$ is the start system and $\mathbf{F}(y) = 0$ is the target system. Here, $\mathbf{G}(y)$ is chosen to be the equations representing configuration 1 of the robot for which closed form solutions are available. $\mathbf{F}(y)$ is the set of equations which represent configuration 2. Using the four solutions (from Table 2) as starting points, solution paths are tracked to determine the inverse solution at the start position (22, 5, 10) for the more general case of configuration 2. The four solutions obtained are listed in Table 3. The factor $e^{i\theta}$ in the homotopy function assures complex paths and thereby avoids any singularities along the path [Wampler et al., 1990b].

Table 2: Inverse kinematic solution for configuration 1

	Soln. 1	Soln. 2	Soln. 3	Soln. 4
θ_1 Rad.	0.004543	-2.699183	0.004543	-2.699183
θ_2 Rad.	-1.20670	2.787378	0.354215	4.348293
θ_3 Rad.	0.036050	0.036050	-3.083594	-3.083594

Table 3: Inverse kinematic solution for configuration 2

	Soln. 1	Soln. 2	Soln. 3	Soln. 4
θ_1 Rad.	-0.363128	3.938811	0.490423	3.091134
θ_2 Rad.	-0.965090	2.7562248	0.2989698	4.373839
θ_3 Rad.	0.172250	0.092693	3.184102	3.259352

TRAJECTORY PLANNING FOR THE 3-R ROBOT

Trajectory planning is accomplished using the continuation method to solve a system of n equations in $(n+1)$ unknowns. For such a system, the solution set will be a family of curves. The objective is to include a variable associated with the trajectory of the robot in cartesian space in solving for the inverse kinematics. A normalized path variable, p, where $p = 0$ at the start of the trajectory, and $p = 1$ at the end was used.

The system of equations for the 3-R robot are Eqs. (3) for configuration 1, and Eqs. (4) for configuration 2. The trajectory is expressed as functions $P_x(p)$, $P_y(p)$ and $P_z(p)$ where, $(P_x(0), P_y(0), P_z(0))$ corresponds to the starting point, and $(P_x(1), P_y(1), P_z(1))$ corresponds to the end point of the trajectory.

Upon substitution of the functions $P_x(p)$, $P_y(p)$ and $P_z(p)$, one can rearrange Eqs. (3) or Eqs. (4) to obtain the following system:

$$\begin{aligned} E_1 &= R.H.S.\ of\ the\ 1st\ of\ Eq. - P_x(p) = 0 \\ E_2 &= R.H.S.\ of\ the\ 2nd\ of\ Eq. - P_y(p) = 0 \\ E_3 &= R.H.S.\ of\ the\ 3rd\ of\ Eq. - P_z(p) = 0 \end{aligned} \tag{7}$$

To solve this system, the extended Jacobian matrix, DE, is formulated by evaluation of the partial derivatives of the functions with respect to the variables θ_1, θ_2, θ_3 and p. The resulting Jacobian is a 3 x 4 matrix and has the form:

$$DE = \begin{bmatrix} \frac{\partial E_1}{\partial \theta_1} & \frac{\partial E_1}{\partial \theta_2} & \frac{\partial E_1}{\partial \theta_3} & \frac{\partial E_1}{\partial p} \\ \frac{\partial E_2}{\partial \theta_1} & \frac{\partial E_2}{\partial \theta_2} & \frac{\partial E_2}{\partial \theta_3} & \frac{\partial E_2}{\partial p} \\ \frac{\partial E_3}{\partial \theta_1} & \frac{\partial E_3}{\partial \theta_2} & \frac{\partial E_3}{\partial \theta_3} & \frac{\partial E_3}{\partial p} \end{bmatrix} \tag{8}$$

This extended Jacobian matrix is used to determine the three first order differential equations of the the joint space variables with respect to the cartesian path variable. Thus,

$$d\theta_i/dp \;=\; (-1)^{i+1} det(DE_{[i]})/Den \tag{9}$$

where, $i = 1, 2, 3$, $DE_{[i]}$ is the Jacobian with the ith column deleted, and Den is the negative determinant of the Jacobian with the fourth column deleted.

The first order differential equations thus obtained are integrated with the known solution at the starting point of the trajectory providing the initial conditions. The integration provides θ_1, θ_2, and θ_3 values as a function of the path variable p. The joint velocity histories are also obtained if the velocity of the manipulator along its trajectory, dp/dt, is specified. The formula $d\theta_i/dt = (d\theta_i/dp)(dp/dt)$ is used to determine the velocities of the joint variables.

Example for configuration 1:

Trajectory planning was carried out for configuration 1 represented by the set of Eqs. (3). The first step was to form the functions $Px(p)$, $Py(p)$ and $Pz(p)$ for the specified trajectory. These functions were then substituted into Eqs. (7) and the extended Jacobian was evaluated as outlined in the previous section. A computer program was developed to generate the first order differential equations numerically which were integrated using both Euler's method and Hamming's predictor corrector method (HPCG) to determine the joint angles along the trajectory.

The procedure was applied to traverse a circle in the cartesian space, starting from the point (22, 5, 10). The radius of the circle used was 5 units and it was situated in the y-z plane. The trajectory functions were expressed as:

$$P_x(p) = 22$$

$$P_y(p) = 5cos(2\pi p)$$

$$P_z(p) = 10 - 5sin(2\pi p)$$

The θ_1, θ_2 and θ_3 starting values were supplied from the inverse solutions in Table 2.

A plot of the resultant joint angles and velocities is given in Fig. 3 for the starting solution 1 as listed in Table 2. The joint velocities plotted were obtained assuming that the manipulator followed its trajectory at a constant unit velocity, i.e., $dp/dt = 1$. On driving the servomotors attached to the joints using the calculated results the robot would trace a circular path with a unit velocity. The trajectory was checked by carrying out the forward kinematic analysis of the robot using a computer and comparing it to the desired path. The results were acceptable except for the Euler's method with a step size of 0.01.

The average CPU time for the HPCG method with a step size (Δp) of 0.01 was 0.4425 seconds and that for a step size of 0.001 was 3.7875 seconds. For Euler's method the CPU time was 1.1425 seconds for a step size of 0.001. All computations were carried out on a VAX 11/785 machine. Computation time could be reduced by using a LU decomposition and back substitution procedure instead of Cramer's rule.

Example for configuration 2:

The inverse kinematic solution was determined for the general system represented by Eqs. (4) for the circular trajectory. Fig. 4 gives the angular position and velocity

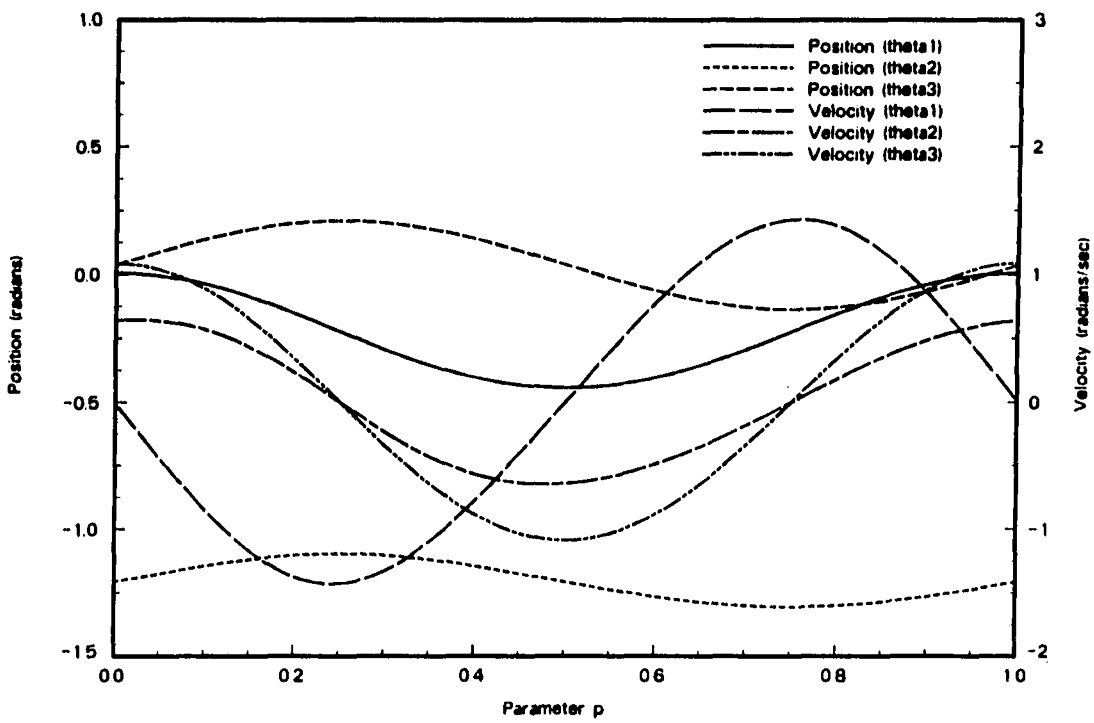

Figure 3: Joint history for configuration 1

information for generating a circular trajectory from $(22, 5, 10)$. The CPU time for obtaining this solution was 0.5325 and 4.82 seconds for HPCG with step sizes of 0.01 and 0.001 respectively, and 1.83 seconds for Euler's method with a Δp value of 0.001. The starting configuration used was the first solution listed in Table 3. All calculations were performed on a VAX/VMS 11/785 machine, as for the previous example.

This configuration represents a general robot for which closed form solutions were not available and so the method could be extremely useful for applications to robots with manufacturing defects or for special robots. The implementation of the method to carry out an on-line control of the trajectory and a study on the effect of path tracking schemes on the CPU time will be the next logical step.

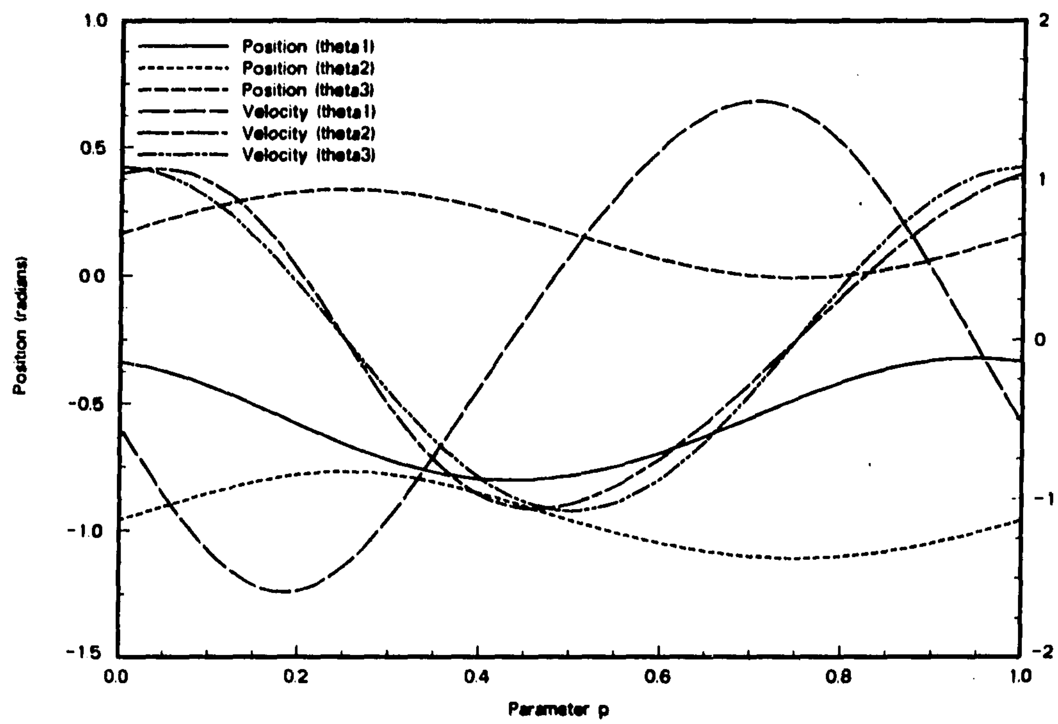

Figure 4: Joint history for configuration 2

TRAJECTORY PLANNING FOR A 6-R PUMA TYPE ROBOT

The transformation matrix for a six degree of freedom robot can be obtained following a procedure similar to that for a 3-R manipulator. The development of this transformation matrix can be obtained from a number of sources [Craig, 1986, Fu et al., 1987 and Wolovich, 1987]. To determine the inverse kinematics for a PUMA robot, 12 equations arising from the position vector and the 3x3 rotation matrix in terms of six unknowns are solved. Only three of the nine elements of the rotation matrix are independent. The procedure outlined by Tsai and Morgan [1985] was utilized to reduce the system to a set of five equations in five unknowns. The final system of equations is:

$$
\begin{aligned}
H_1 &= C\theta_1 C\theta_2 n_x + S\theta_1 C\theta_2 n_y - S\theta_2 n_z + C\theta_3 C\theta_4 S\theta_5 + S\theta_3 C\theta_5 = 0 \\
H_2 &= -S\theta_1 n_x + C\theta_1 n_y - S\theta_4 S\theta_5 = 0 \\
H_3 &= C\theta_1 C\theta_2 P_x + S\theta_1 C\theta_2 P_y - S\theta_2 P_z - a_2 + S\theta_3 d_4 - a_3 C\theta_3 = 0 \\
H_4 &= -C\theta_1 S\theta_2 P_x - S\theta_1 S\theta_2 P_y - C\theta_2 P_z - C\theta_3 d_4 - a_3 S\theta_3 = 0 \\
H_5 &= -S\theta_1 P_x + C\theta_1 P_y - d_3 = 0 \qquad (10)
\end{aligned}
$$

In the above equations, lengths a_2, a_3, d_3, d_4 and angles θ_1, θ_2 , θ_3 are as defined for configuration 1 of the 3-R robot (Table 1). Angles θ_4, θ_5 and θ_6 are the angular orientations of the wrist, **n** is the unit vector attached to the z axis of the final coordinate system and **P** is the position vector of the point where the three joint axes of the wrist intersect.

For a trajectory planning problem, the position vector **P** and the orientation vector **n** are specified along the path, and we must determine the joint space variable values, θ_1, θ_2, θ_3, θ_4 and θ_5 from Eqs. (10). Using the angles θ_1 through θ_5 and the remaining elements of the rotation matrix, one can calculate θ_6.

To determine the joint space histories, it can be assumed that the angular orientation and velocity are known at the initial point on the trajectory. Path tracking is carried out using continuation as described for a 3-R manipulator. For the set of equations under consideration, the extended Jacobian matrix, DH, is a 5 x 6 matrix. The derivatives of the joint space variables with respect to the cartesian path variable p are obtained using DH in Eq. (9), instead of DE.

Example 3: 6-R PUMA type robot

The robot was to travel along a circle from (22, 5, 10), and the z axis of the coordinate system attached to the end-effector was forced to maintain a 45^o angle with the trajectory path. Since the angular orientation of the z axis is fixed, the 3x3 rotation matrix will vary depending on the location of the end-effector along the circle. This type of motion is frequently encountered in welding applications.

The overall transformation matrix has the following form:

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 22 \\ -cos(2\pi p)/\sqrt{2} & -sin(2\pi p) & cos(2\pi p)/\sqrt{2} & 5cos(2\pi p) \\ sin(2\pi p)/\sqrt{2} & -cos(2\pi p) & -sin(2\pi p)/\sqrt{2} & 10 - 5sin(2\pi p) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

The first three elements of the last column are $P_x(p)$, $P_y(p)$ and $P_z(p)$ respectively and that of the third column are $n_x(p)$, $n_y(p)$ and $n_z(p)$ respectively. These values were substituted into Eqs. (10), and the inverse kinematic solution along the path was evaluated. The joint space displacement and velocity histories are shown in Figs. 5 and 6. The CPU time spent in computing the solution was 1.15 sec and 10.3

sec for the HPCG method with step sizes of 0.01 and 0.001 respectively and 4 sec for Euler's method with a step size of 0.001.

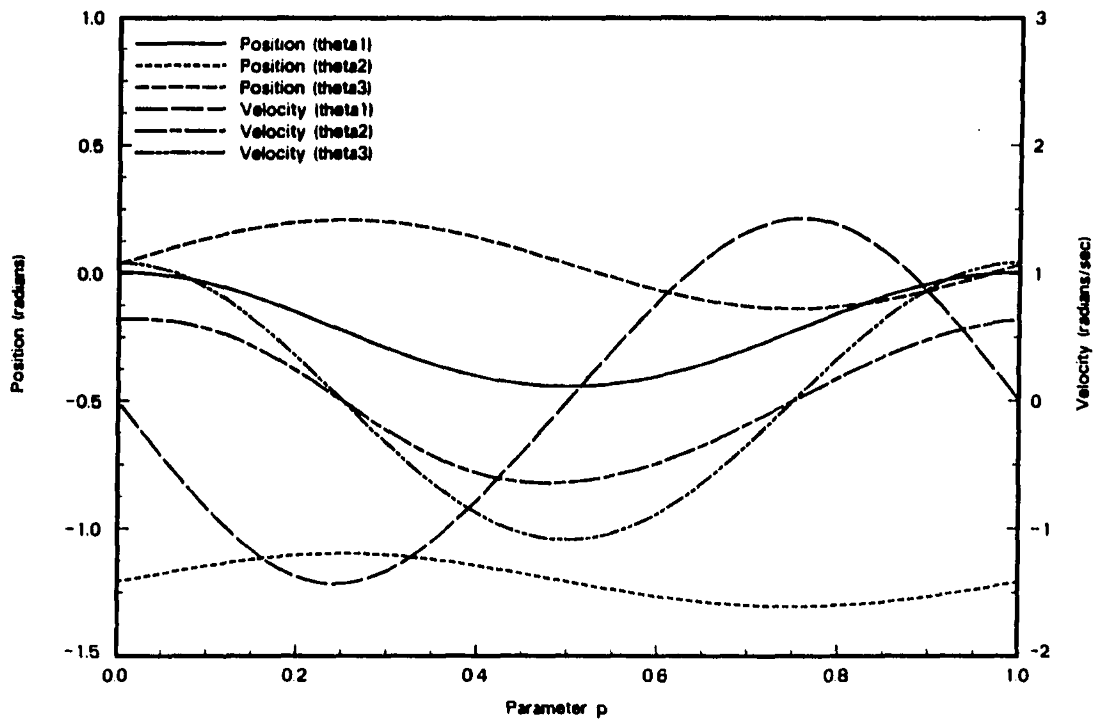

Figure 5: History of joint variables θ_1, θ_2 and θ_3 for 6-R PUMA robot

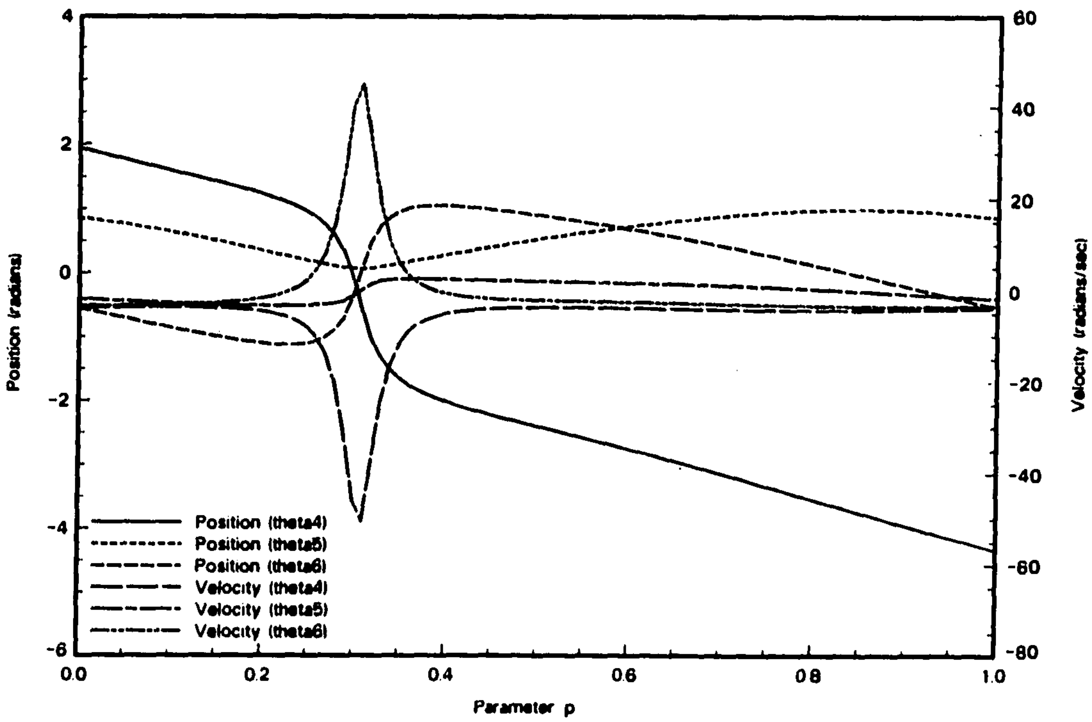

Figure 6: History of joint variables θ_4, θ_5 and θ_6 for 6-R PUMA robot

REFERENCES

Angeles, J. 1986. Iterative Kinematic Inversion of General Five-Axis Robot Manipulators. The International Journal of Robotics Research 4, no.4:59-70.

Chand, S., and K. L. Doty. 1985. On-Line Polynomial Trajectories for Robot Manipulators. The International Journal of Robotics Research 4, no.2:38-48.

Craig, J. J. 1986. Introduction to Robotics. Addison-Wesley Publishing Company, Reading, Massachusetts.

Fu, K. S., R. C. Gonzalez, and C. S. G. Lee. 1987. Robotics: Control, Sensing, Vision, and Intelligence. McGraw-Hill Book Company, New York, New York.

Goldenberg, A. A., and D. L. Lawrence. 1986. End Effector Path Generation. Journal of Dynamic Systems, Measurement, and Control 108-2:158-162.

Gupta, K. C., and K. Kazerounian. 1985. Improved Numerical Solutions of Inverse Kinematics of Robots. IEEE International Conference on Robotics and Automation 1985: 743-748.

Hornick, M. L., and B. Ravani. 1986. Computer-Aided Off-Line Planning and Programming of Robot Motion. The International Journal of Robotics Research 4, no.4:18-31.

Jo, D. -Y., and E. J. Haug. 1988. Workspace Analysis of Multi body Mechanical Systems using Continuation Methods. pp 447-456. Trends and Developments in Mechanisms, Machines and Robotics. The American Society of Mechanical Engineers, New York, New York.

Morgan, A. 1981. A Method for Computing all Solutions to Systems of Polynomial Equations. GM Research Publication, GMR 3651.

Morgan, A. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Morgan, A. P., and C. W. Wampler. 1989. Solving a Planar Four-Bar Design Problem using Continuation. Advances in Design Automation - 1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:409-416.

Pickett, M. S., and R. M. Zarger. 1983. Off-Line Robot Programming in GMSOLID: A Feasibility Study. GM Research Publication, GMR 4484.

Singh, V. K. and K. C. Gupta. 1989. A Manipulator Jacobian Based Modified Newton-Raphson Algorithm (JMNR) for Robot Inverse Kinematics. Advances in Design Automation - 1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:327-332.

Starns, K. G., and D. R. Flugrad. 1990. Five-Bar Path Generation Synthesis by Continuation Methods. Mechanism Synthesis and Analysis. ASME DE-25:353-359.

Subbian, T., and D. R. Flugrad. 1989a. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:425-432.

Subbian, T., and D. R. Flugrad. 1989b. Synthesis of Motion Generating Mechanisms by a Continuation Method. 1st National Applied Mechanisms and Robotics Conference 89AMR-2B-2.

Subbian, T., and D. R. Flugrad. 1990. Five Position Triad Synthesis With Applications to Four- and Six-Bar Mechanisms. Accepted for publication in the Journal of Mechanical Design.

Taylor, R. H. 1979. Planning and Execution of Straight Line Manipulator Trajectories. IBM Journal of Research and Development 23, no.4:424-436.

Tsai, L. -W., and J. J. Lu. 1989. Coupler-Point-Curve Synthesis Using Homotopy Methods. Advances in Design Automation-1989: Mechanical Systems Analysis, Design and Simulation. ASME DE-19-3:417-423.

Tsai, L. -W., and A. P. Morgan. 1985. Solving the Kinematics of the most General Six- and Five-Degree-of-Freedom Manipulators by Continuation Methods. Journal of Mechanisms, Transmissions and Automation in Design 107:189-200.

Wampler, C., and A. Morgan. 1989. Solving the 6-R Inverse Position Problem Using a Generic-Case Solution Methodology. GM Research Publication, GMR 6702.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990a. Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages. Mechanism Synthesis and Analysis. ASME DE-25:361-368.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990b. Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics. Journal of Mechanical Design 112, no.1:59-68.

Wang, K. 1988. B-Splines Joint Trajectory Planning. Computers in Industry 10, no.2:113-122.

Wolovich, W. A. 1987. Robotics: Basic Analysis and Design. CBS College Publishing, New York, New York.

CONCLUSIONS

Continuation methods are a very robust mathematical tool for solving kinematic synthesis problems. They provide all the solutions for systems of polynomial equations and unlike the currently popular methods, they do not require any initial solution estimates. Kinematic equations can often be expressed as nonlinear polynomial systems and continuation method is particularly useful to solve analysis and synthesis problems. The ability of continuation method to solve n equations in $(n+1)$ unknowns can be effectively used to trace solution curves (Burmester curves) there by providing infinite solutions as opposed to a finite solution set obtained by other methods.

This dissertation considered continuation methods for the kinematic synthesis and analysis of mechanisms. Four-bar five position path generating mechanism was designed, for the first time. Wampler et al. [1990] extended the number of path points to nine using continuation method. To the best of our knowledge, the seven position triad synthesis for motion generation with prescribed timing was solved for the first time. For the five and six position triad design, solution curves were traced to give infinities of triads. All the solutions for the four positions revolute-spherical dyad were obtained, and a continuation approach to robot trajectory planning was presented for the first time.

Planar four-bar motion, function and path generation mechanisms were designed. The motion generation problem capitalized on the ability of the continuation method to solve n equations in n as well as $(n + 1)$ unknowns. A complete solution set for a five position path generation problem was obtained.

Triad synthesis is important since single and multi-loop mechanisms can be broken up into dyads and triads and the individual components can be designed separately. Synthesis of triads for five, six and seven prescribed positions for motion generation with timing were considered. These triads were used to design the geared five-bar, six-bar and eight-bar mechanisms. In this dissertation, the ability of the continuation method to trace solution curves was utilized to generate Burmester curves. The fixed pivot, coupler pivot and intermediate pivot locations were plotted. Lin and Erdman [1987] solved the six position problem using a compatibility linkage approach and obtained the triad Burmester curves. Their technique is quite involved while the approach used in this dissertation reduces the problem to a simple path tracking problem. The seven position synthesis problem was solved using a secant homotopy which requires the tracking of just 17 paths.

A step by step procedure to implement continuation methods with projective transformation and without the coefficient independent solutions at infinity was presented in part V of the dissertation. The seven position triad synthesis problem and a spatial revolute-spherical dyad synthesis problem were considered to demonstrate the concept.

Trajectory planning of robots by continuation methods gave joint displacement and velocity information continuously along the path. The ability of the method to solve n equations in $(n + 1)$ unknowns was used to solve this problem where the path variable was made the $(n+1)$th variable. The method shows good potential for online implementation in robot controllers.

Although this dissertation is restricted mainly to planar mechanisms (except for the R-S dyad), the procedure could easily be extended to solve spatial synthesis and analysis problems. Future work can be focused on the application of the procedure to design complex spatial mechanisms like the RRSS and RSSR-SS type mechanisms for path and motion generation. Also, the spatial dyads can be designed and the Burmester curves generated, similar to those for planar dyad synthesis. Optimization of the design (both planar and spatial) and order synthesis can also be considered.

The potential of continuation methods is immense. With the developments in computer technology and the advancements in the method, kinematic synthesis problems which seemed impossible can be effectively solved. This dissertation has focused on testing out the procedure and evaluating its usefulness. We conclude that continuation methods have excellent features when compared to the currently used numerical procedures and is likely to play an important role in kinematic synthesis and analysis in future.

REFERENCES

Erdman, A. G. 1981. Three and Four Point Kinemtic Synthesis of Planar Linkages. Journal of Mechanism and Machine Theory 16:227-245.

Garcia, C. B., and W. I. Zangwill. 1981. Pathways to Solutions, Fixed Points, and Equilibria. Prentice-Hall, Englewood Cliffs, New Jersey.

Kramer, S. N., and G. N. Sandor. 1975. Selective Precision Synthesis-A General Method of Optimization for Planar Mechanisms. Journal of Engineering for Industry97B, no.2:689-701.

Lin, C. -S., and A. G. Erdman. 1987. Dimensional Synthesis of Planar Triads: Motion Generation With Prescribed Timing for Six Precision Positions. Journal of Mechanism and Machine Theory 22:411-419.

Morgan, A. P. 1986a. A Homotopy for Solving Polynomial Systems. Applied Mathematics and Computation 18:87-92.

Morgan, A. P. 1986b. A Transformation to Avoid Solutions at Infinity for Polynomial Systems. Applied Mathematics and Computation 18:77-86.

Morgan, A. P. 1987. Solving Polynomial Systems using Continuation for Engineering and Scientific Problems. Prentice-Hall, INC., Englewood Cliffs, New Jersey.

Morgan, A. P., and A. J. Sommese. 1987a. Computing All Solutions to Polynomial Systems Using Homotopy Continuation. Applied Mathematics and Computation 24:115-138.

Morgan, A. P., and A. J. Sommese. 1987b. A Homotopy for Solving General

Polynomial Systems That Respects m-Homogeneous Structures. Applied Mathematics and Computation 24:101-113.

Morgan, A. P., and A. J. Sommese. 1989. Coefficient-Parameter Polynomial Continuation. Applied Mathematics and Computation 29:123-160.

Roth, B., and F. Freudenstein. 1963. Synthesis of Path-Generating Mechanisms by Numerical Methods. ASME Journal of Engineering for Industry. 85B, on.3:298-306.

Sandor, G. N., and A. G. Erdman. 1984. Advanced Mechanism Design: Analysis and Synthesis. Vol 2. Prentice-Hall, Englewood Cliffs, New Jersey.

Sarkisyan, Y. L., K. C. Gupta, and B. Roth. 1973. Kinematic Geometry Associated with the Least-Square Approximation of a Given Motion. Journal of Engineering for Industry 95, no.2:503-510.

Subbian, T., and D. R. Flugrad. 1989a. Synthesis of Motion Generating Mechanisms by a Continuation Method. 1st National Applied Mechanisms and Robotics Conference 89AMR-2B-2.

Subbian, T., and D. R. Flugrad. 1989b. Four-Bar Path Generation Synthesis by a Continuation Method. Advances in Design Automation-1989: Mechanical Syatems Analysis, Design and Simulation. ASME DE-19-3:425-432.

Subbian, T., and D. R. Flugrad. 1990a. Five Position Triad Synthesis with Applications to Four- and Six-Bar Mechanisms. Accepted for publication in ASME Journal of Mechanical Design.

Subbian, T., and D. R. Flugrad. 1990b. Six and Seven Position Triad Synthesis Using a Continuation Method. Under review for publication in the ASME Journal of Mechanical Design.

Subbian, T., and D. R. Flugrad. 1991. Use of Continuation Methods for Kinemtic Synthesis. Submitted to the 8th World Congress on the Theory of Machines and Mechanisms, Prague, Czechoslovakia, August 1991.

Subbian, T., D. R. Flugrad, and P. Kavanagh. 1991. Robot Trajectory Planning by a Continuation Method. Accepted for presentation at the Joint International Conference, FAIM 91, Factory Automation and Information Management, Limerick, Ireland, March 1991.

Suh, C. H., and C. W. Radcliffe. 1978. Kinematics and Mechanism Design. John Wiley and Sons, Inc., New York.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990a. Numerical Continuation Methods for Solving Polynomial Systems Arising in Kinematics. Journal of Mechanical Design 112, no.1:59-68.

Wampler, C. W., A. P. Morgan, and A. J. Sommese. 1990b. Complete Solution of the Nine-Point Path Synthesis Problem for Four-Bar Linkages. Mechanism Synthesis and Analysis. ASME DE-25:361-368.

APPENDIX A: MULTI-HOMOGENEOUS STRUCTURE AND PROJECTIVE TRANSFORMATION

The major feature of the multi-homogeneous (m-homogeneous) system is that the homotopy may be chosen to closely resemble the system being solved. The choice of such homotopies reduces the number of paths and has the potential of reducing the arc length of the paths. The m-homogeneous structure is discussed here due to its significant advance over the traditional 1-homogeneous homotopy, and also since many systems in kinematics have such structures. A detailed description of the principles described here can be found in Morgan and Sommese [1987b].

A multivariable polynomial $F(z) = 0$ is m-homogeneous if there are m sets of variables

$$Z_j = \{z_{0j}, z_{1j}, ..., z_{k_j j}\}$$

for $j = 1$ to m and m nonnegative integers d_1, d_2, ..., d_m such that

$$F(z) = \sum_{|I_j|=d_j; j=1,m} a_{(I_1,I_2,...,I_m)} Z_1^{I_1} ... Z_m^{I_m},$$

where $I_j = \{i_{0j}, i_{1j}, ..., i_{k_j j}\}$ is an index set, "$|I_j| = d_j$" means "$\sum_{s=1}^{k_j} i_{sj} = d_j$," and

$$Z_j^{I_j} = z_{0j}^{i_{0j}} z_{1j}^{i_{1j}} ... z_{k_j j}^{i_{k_j j}}.$$

The $a_{(I_1,I_2,\ldots,I_m)}$ are complex coefficients and z_{0j}s are homogeneous variables.

Consider next a system, $\mathbf{F}(z) = 0$, of n m-homogeneous polynomials. If $d_{j,l}$ denotes the degree of the jth set of variables of the lth equation of the system, the Bezout number of $\mathbf{F}$ is defined as the integer d which is the coefficient of $\prod_{j=1}^{m} \alpha_j^{k_j}$ in the combinatorial product

$$\prod_{l=1}^{n} \sum_{j=1}^{m} d_{j,l}\alpha_j$$

Bezout's theorem

Let $\mathbf{F}(z) = 0$ be an m-homogeneous system as defined above. Then $\mathbf{F}(z) = 0$ has no more than d geometrically isolated solutions. If $\mathbf{F}(z) = 0$ does not have an infinite number of solutions, then it has exactly d solutions, counting multiplicity.

Example 1: 2-homogeneous system

Consider the system,

$$\begin{aligned} F_1(z) &= C_{1,1}z_1z_2 + C_{1,2}z_1 + C_{1,3}z_2 + C_{1,4} = 0 \\ F_2(z) &= C_{2,1}z_1^2 + C_{2,2}z_1 + C_{2,3} = 0 \end{aligned} \tag{1}$$

We define the sets Z_1 and Z_2 according to

$Z_1 = \{z_{01}, z_{11}\}$

$Z_2 = \{z_{02}, z_{12}\}$

The substitution $z_1 \leftarrow z_{11}/z_{01}$ and $z_2 \leftarrow z_{12}/z_{02}$ is used to 2-homogenize the system of equations. On simplifying:

$$\begin{aligned} F_1(z) &= C_{1,1}z_{11}z_{12} + C_{1,2}z_{11}z_{02} + C_{1,3}z_{12}z_{01} + C_{1,4}z_{01}z_{02} = 0 \\ F_2(z) &= C_{2,1}z_{11}^2 + C_{2,2}z_{11}z_{01} + C_{2,3}z_{01}^2 = 0 \end{aligned} \tag{2}$$

For the above equations,

$$\begin{aligned} d_{1,1} &= \text{ degree of 1st set in 1st equation } = & 1 \\ d_{2,1} &= \text{ degree of 2nd set in 1st equation } = & 1 \\ d_{1,2} &= \text{ degree of 1st set in 2nd equation } = & 2 \\ d_{2,2} &= \text{ degree of 2nd set in 2nd equation } = & 0 \end{aligned}$$

and the Bezout number of the system is the coefficient of $\alpha_1\alpha_2$ in the product $[(\alpha_1 + \alpha_2)(2\alpha_1 + 0\alpha_2)]$, which is 2 for the present example.

Example 2: 3-homogeneous system

Consider the system

$$\begin{aligned} F_n(z) &= A_{1,n}z_1z_5 + A_{2,n}z_2z_5 + A_{3,n}(z_2z_3 - z_1z_4) + A_{4,n}(z_1z_3 + z_2z_4) \\ &\quad + A_{5,n}z_3z_5 + A_{6,n}z_4z_5 + A_{7,n}z_1 + A_{8,n}z_2 + A_{9,n}z_3 + A_{10,n}z_4 \\ &\quad + A_{11,n}z_5 + A_{12,n} = 0 \end{aligned} \tag{3}$$

where $n = 1$ to 5. We form three sets of variables as follows:

$$Z_1 = \{z_{01}, z_{11}, z_{21}\}$$

$$Z_2 = \{z_{02}, z_{12}, z_{22}\}$$

$$Z_3 = \{z_{03}, z_{13}\}$$

The following substitutions are made to homogenize the system of equations: $z_1 \leftarrow z_{11}/z_{01}$; $z_2 \leftarrow z_{21}/z_{01}$; $z_3 \leftarrow z_{12}/z_{02}$; $z_4 \leftarrow z_{22}/z_{02}$ and $z_5 \leftarrow z_{13}/z_{03}$ The homogeneous system of equations obtained is:

$$
\begin{aligned}
F_n(z) = {} & A_{1,n} z_{11} z_{13} z_{02} + A_{2,n} z_{21} z_{13} z_{02} + A_{3,n}(z_{21} z_{12} - z_{11} z_{22}) z_{03} \\
& + A_{4,n}(z_{11} z_{12} + z_{21} z_{22}) z_{03} + A_{5,n} z_{12} z_{13} z_{01} + A_{6,n} z_{22} z_{13} z_{01} \\
& + A_{7,n} z_{11} z_{02} z_{03} + A_{8,n} z_{21} z_{02} z_{03} + A_{9,n} z_{12} z_{01} z_{03} \\
& + A_{10,n} z_{22} z_{01} z_{03} + A_{11,n} z_{13} z_{01} z_{02} + A_{12,n} z_{01} z_{02} z_{03} = 0 \qquad (4)
\end{aligned}
$$

In Eqs. (4), $d_{1,n} = 1$; $d_{2,n} = 1$; $d_{3,n} = 1$ and the Bezout number is the coefficient of $(\alpha_1^2 \alpha_2^2 \alpha_3)$ in the product $(\alpha_1 + \alpha_2 + \alpha_3)^5$, which is 30.

In order to take advantage of the path reduction offered by the m-homogeneous structure of the target system, the start system should also be of the same structure. A m-homogeneous coefficient, secant or parameter homotopy can be used to solve these equations.

Projective transformation

The identification of m-homogeneous structures eliminates a few of the solutions at infinity, but not all. Thus computer codes must decide if a path is diverging or not and abort the paths that diverge. Solutions at infinity result in more computation time and also present a dilemma as to when to declare that a path is diverging. This decision cannot be made without the risk of truncating a converging path. Therefore, techniques which transform solutions at infinity to finite solutions are essential [Morgan and Sommese, 1987b and Morgan 1986b].

The m homogeneous variables (z_{0j}; j=1 to m) introduced in the previous subsection homogenize the n target equations. On carrying out the homogenization we end up with n equations in $(n+m)$ unknowns, as given by Eqs. (2) and (4). Certain formulae are included which make the z_{0j}'s implicitly defined functions of the other k_j variables in the same set as follows:

$$z_{0j} = \sum_{i=1}^{k_j} \beta_{ij} z_{ij} + \beta_{0j} \tag{5}$$

for $j = 1$ to m, where β's are random complex constants.

We can use Eqs. (5) to eliminate the m homogeneous variables from the homotopy functions to obtain n homogeneous equations in n unknowns. We can otherwise include the m equations with the n homotopy functions and solve for $(n+m)$ unknowns. This does not increase the Bezout number, as Eqs. (5) are all of degree 1. On solving, we can recover the solutions of $\mathbf{F}(z) = 0$ through the transformation $z_n \leftarrow z_{ij}/z_{0j}$. If $\mathbf{F}(z) = 0$ has a solution at infinity, at least one of the m z_{0j} values will be 0.

In example 1 (Eq. (1)) 2 homogeneous variables, z_{01} and z_{02}, were used to homogenize the equations. For projective transformation we include the following formulae:

$$\begin{aligned} z_{01} &= \beta_{11} z_{11} + \beta_{01} \\ z_{02} &= \beta_{12} z_{12} + \beta_{02} \end{aligned} \tag{6}$$

We can solve Eqs. (2) and (6) together for the unknowns z_{01}, z_{02}, z_{11} and z_{12}. The transformations $z_1 = z_{11}/z_{01}$ and $z_2 = z_{12}/z_{02}$ are used to recover the original variables. For Example 2 the homogenization variables are defined as follows:

$$\begin{aligned} z_{01} &= \beta_{11} z_{11} + \beta_{21} z_{21} + \beta_{01} \\ z_{02} &= \beta_{12} z_{12} + \beta_{22} z_{22} + \beta_{02} \\ z_{03} &= \beta_{13} z_{13} + \beta_{03} \end{aligned} \tag{7}$$

APPENDIX B: COEFFICIENT, SECANT AND PARAMETER HOMOTOPIES

If the system of equations $\mathbf{F}(c[q], x) = 0$, where $c[q]$ are the coefficients of the equations and q are the parameters the coefficients depend on, is to be solved for just one set of parameters, a 1-homogeneous traditional homotopy would be sufficient. But in many applications, including kinematics, we would repeatedly solve the same set of equations for different choice of parameters. This calls for a modified homotopy.

The modified homotopy uses a start system which is of the same structure as the system of equations to be solved. The solutions for this start system are obtained using a traditional 1-homogeneous homotopy or a traditional coefficient homotopy depending on whether the system of equations is 1-homogeneous or m-homogeneous. These solutions are sorted according to whether they are singular or nonsingular, and as to whether they are solutions at infinity or they are finite solutions. From the entire solution set, the solutions having the properties desired in the solutions of the target system are selected. Those solutions comprise the set of nonsingular finite solutions. Such solutions of the start system are used to implement the modified homotopies.

The first step is to obtain the solutions for the start system, which resemble the target system we want to repeatedly solve. Random parameter values (q^0's) are substituted into the given system of equations. These equations, $\mathbf{F}(c[q^0], x) = 0$, are solved using a generic start system of the form: $G_j(x) = p_j^{d_j} - r_j^{d_j} = 0$, if the given equation is of 1-homogeneous form. Otherwise, a simple m-homogeneous system is used to solve the target system. Once the solutions are obtained, we sort them as before and use the desired solutions of the system $\mathbf{F}(c[q^0], x) = 0$ to solve future problems.

Coefficient homotopy

The coefficient homotopy described here is different from the m-homogeneous traditional coefficient homotopy. In traditional coefficient homotopy, the solutions of the start system should all be nonsingular and the equations easily solvable. Here, however, the solutions of the start system can be singular and are as complicated to solve as the target system. A typical coefficient homotopy [Morgan and Sommese, 1989] which solves $\mathbf{F}(c[q^k], x) = 0$ is as follows:

$$\mathbf{H}(x,t) = \mathbf{F}(((1-t)C^0 + tc[q^k]), x) \qquad (1)$$

where, C^0's are random complex coefficients and $c[q^k]$ are the coefficients for the specified parameters q^k's (displacements in kinematic synthesis). Using the solutions of the system of equations for randomly chosen coefficients (C^0's), the system of equations for an arbitrary set of parameters can be solved.

Coefficient homotopy is easy to program and is cheaper in terms of function and Jacobian evaluations. The number of paths to be tracked will be less than or equal to that of the traditional coefficient homotopy but will never be less than the secant or parameter homotopies. Since the coefficients of the start system are complex, the paths will be in the complex space, and thereby singularities will be avoided.

Parameter homotopy

In coefficient homotopy, though the coefficients are a function of only a few parameters, each coefficient is chosen independently. This increases the number of free choices, and thus the start system does not resemble the system of equations being solved. Therefore, a parameter homotopy can be used which needs random selection of a few parameters. Parameter homotopy, besides acknowledging the specific structure of the target system, has fewer paths to be tracked.

A typical parameter homotopy is given by the following expression [Morgan and Sommese, 1989]:

$$\mathbf{H}(x,t) = \mathbf{F}(c[(1-t)q^0 + tq^k], x) \tag{2}$$

where q^0 must be chosen randomly, and q^k are the specified parameters.

Analytically determining the Jacobian and programming this homotopy is difficult compared to the coefficient or secant homotopies. Also, the CPU time to track one path is increased significantly. However, when solving large problems, side conditions that the solutions of the target system should satisfy are imposed. These side conditions can be effectively implemented with the parameter homotopy, and so this homotopy results in the least number of paths.

Secant homotopy

The secant homotopy is a compromise between the above mentioned homotopies. Due to its ease of programming, cheaper Jacobian and functional evaluations, and its ability to acknowledge the structure of the equations being solved, the secant homotopy may be the best choice in many cases. The number of paths to be tracked to determine the solution set will be less than that of coefficient homotopy but might be greater than that of parameter homotopy. Depending on the problem, we can cut down the number of paths to that number required for parameter homotopy.

For secant homotopy, the parameters are chosen randomly (not necessarily complex) and the coefficients are evaluated from these parameters. The homotopy function is written connecting the coefficients as follows [Wampler et al., 1990a]:

$$\mathbf{H}(x,t) = \mathbf{F}(((1-t)\gamma c[q^0] + tc[q^k]), x) \tag{3}$$

where, q^0, q^k are as defined before and $\gamma = e^{i\theta}$, θ being a random angle. The factor γ is used to make the paths complex, and therefore, to avoid singularities.

ACKNOWLEDGEMENTS

I will forever be deeply indebted to Dr. Donald R. Flugrad for educating me and guiding me in the pursuit of this research work. I thank him for the countless hours he spent discussing, directing, reading, correcting my work, and for teaching me to be a better student and inspiring me to be a better teacher.

I thank Dr. Patrick Kavanagh for introducing me to continuation methods and for reading and correcting my dissertation to make it a better piece of work. The suggestions and directions which came out of the weekly meetings are gratefully acknowledged.

My thanks to Dr. James E. Bernard for serving on the committee as a co major professor, and supporting me throughout my stay here at Iowa state University without which none of this would have been possible. I thank Dr. Jeffrey C. Huston and Dr. Bion L. Pierson for serving on my committee and for taking the time to read through my dissertation.

I owe everything that I have achieved until now to my parents and family. Without their blessings, love, encouragement and support none of this would have been possible. I give them my deepest love and thanks. The amount of support and encouragement provided by my brother, Dr. Govindaraj, and sister-in-law, Nalini, cannot be put in words. I will ever be indebted to them for the help they provided.

My thanks go to Chandrakumar who is the greatest friend one could possibly have.

I appreciate the inspiration and encouragement provided by Gloria Starns. I thank her for building up the confidence in me and giving me an opportunity to share my research experience.

My thanks to Anand and Priya, Andy and Ranjani, Giri, Kumar, Molians, Ramu, and Srikanth(s), for their help and support. They and my many other friends make life enjoyable, just knowing them and sharing it with them. I thank them for making my long stay in Ames pleasant, happy and unforgettable. And finally, thanks to the lunch group - Brian, Bruce, Gaylord and Peri for the wonderful company.